On the Motion of a Vortex Ring and a Vortex Pair*

- Have we had a centurial misunderstanding? -

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Abstract. To examine the engineering application of an artificial vortex ring, the induced velocity field around a vortex ring has been studied with the potential flow theory. As results, since the induced velocity becomes zero far apart from the vortex ring, the ring does not move by its own induced flow field. In other words, a vortex ring does not necessarily move at the speed as usually it might have been misunderstood.

In this paper, the flow field induced by a vortex ring and then superimposing various strength uniform flows on the field are calculated and discussed. This paper also clears that two parallel straight vortex lines neighboring to each other do not necessarily rotate or go parallel with the velocities, which may have been misunderstood for longtime.

Keywords: Potential flow theory, Vortex ring, Vortex pair, Induced velocity, Vortex motion.

1. INTRODUCTION

One of the authors has recently inspired the usage of artificial vortex ring for various fluid machinery devices [1]. Artificial vortex ring may serve as a new wind fan for ventilation. When it is set at the outlet of a diffuser, vortex ring might improve the diffuser efficiency at considerable rate.

Before the application of the artificial vortex ring, authors examined the induced flow field by a vortex ring with the potential flow theory. Then, a problem of the necessary force to fix the vortex ring to the space was turned out. How large force would be needed to fix the ring artificially rotating around its circumferential vortex line? In other words, if the vortex ring is not fixed to space, the ring shall move the way just as any conventional textbook has taught us for long time.

In a fluid dynamics textbook, motions of two parallel straight vortex lines are also described as if they are moved relatively to each other at the velocity uniquely decided by their relative strength of vortices. In this paper, authors' calculations and discussions have focussed these basic problems of vortex rings and vortex pairs.

NOMENCLATURE

$d_{1,2}$: minimum, maximum length from vortex ring
D : length from vortex ring
E : second complete elliptic integral
F : first complete elliptic integral
$r_{1,2}$: length from each vortex line
R : radius of vortex ring
U : uniform flow velocity
U_{crl} : critical velocity
x, y, z : Cartesian coordinates
X, Y: position of vortex pair
z, r, φ: cylindrical coordinates
Z, R : position of vortex ring
(Greek letters)
ε : radius of a circular vortex core
ψ : stream function
Γ : circulation
λ : $=(d_2-d_1)/(d_2+d_1)$
π : ratio of the circumference

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2. THEORY AND DISCUSSION

2.1 Case of a stationary vortex ring

Potential flow field around a space fixed vortex ring is obtained as in Equation (1) and in Figure 1.

$$\psi = \frac{\Gamma R r}{4\pi}\int_0^{2\pi}\frac{\cos\varphi}{D}d\varphi = \frac{\Gamma}{4\pi}(d_1+d_2)[F(\lambda)-E(\lambda)] \quad (1)$$

$$D = \left[(z-Z)^2 + r^2 + R^2 - 2rR\cdot\cos\varphi\right]^{1/2}$$

$$\lambda = \frac{d_2-d_1}{d_2+d_1} \qquad d_{1,2} = \left[(z-Z)^2 + (r \mp R)^2\right]^{1/2}$$

Figure 1 shows streamlines in the central plane including the central flow axis. As seen from Equation 1 and the figure, the flow velocity far apart from the vortex ring becomes zero, so that entire flow field is at rest to the vortex ring. In other words the vortex ring is fixed to the space, evidently because the calculation has been practiced about "a space fixed vortex ring".

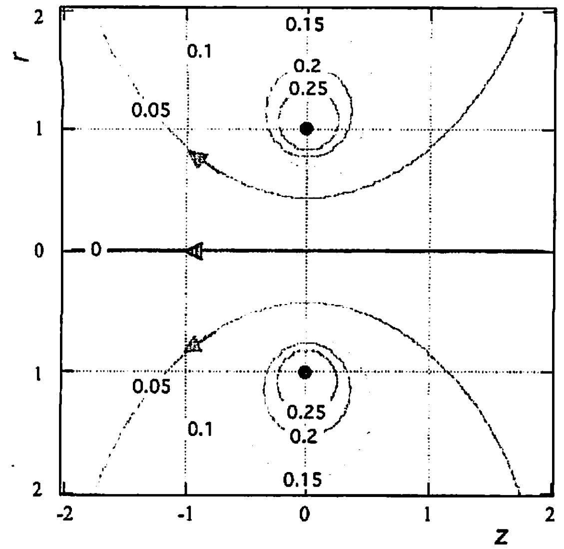

Fig.1 Stream line around a stationary vortex ring
(R=1, Γ=1, U=0)

The induced flow speed at the center of the figure is calculated as following Equation (2). We may call this speed as the critical speed, because this speed may become the key speed in the case when the vortex ring is moving relatively to the space.

$$U = \frac{\Gamma}{2R} = U_{cri} \qquad \text{(Critical speed)} \qquad (2)$$

In the normalized flow field(R=1, Γ=1) in Figure 1, the critical speed is calculated as 0.5.

2.2 Cases of several moving vortex rings

Figure 2 shows the superimposing of a uniform flow on the vortex ring inducing flow indicated in Figure 1, and the uniform flow strength is one fourth of the critical speed i.e. 0.125. While the figure is traced on the coordinates fixed to the vortex ring, we can see the figure as the flow around a slowly moving vortex ring relatively to the space. In this case, the flow accompanied with the vortex ring is circulating only inside of a spherical area. One has to notice that it is not so called Hill's spherical vortex flow, because in the Hill's vortex the vortices are uniformly distributed in the spherical area.

Figure 3 shows second case of a moving vortex ring, in which the superimposing uniform flow strength is exactly the critical. The figure is observed by the coordinates fixed to the vortex ring that is moving at the critical speed relatively to the space. Evidently, flow speed at the center of the figure becomes zero, since the uniform flow and the vortex inducing flow are cancelled each other at the center. In this case, the outer flow can never go through or penetrate the inner field of the vortex ring. The superimposing uniform flow is the same speed but has opposite direction to the induced velocity. So that, we may call this speed as "critical". It is cleared that if the superimposing uniform flow is larger than the critical speed the outer flow can go through the inside of the vortex ring. If the speed is week, outer flow can not go through in the ring and the fluid around the vortex ring can not go out from the near field.

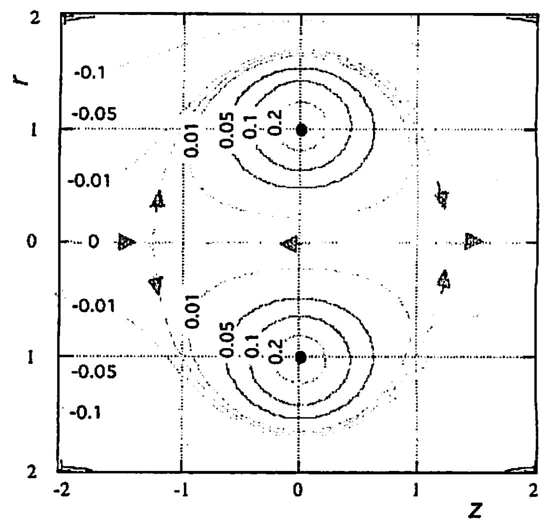

Fig.2 Streamlines around a slowly moving vortex ring
(R=1, Γ=1, U= 0.125)

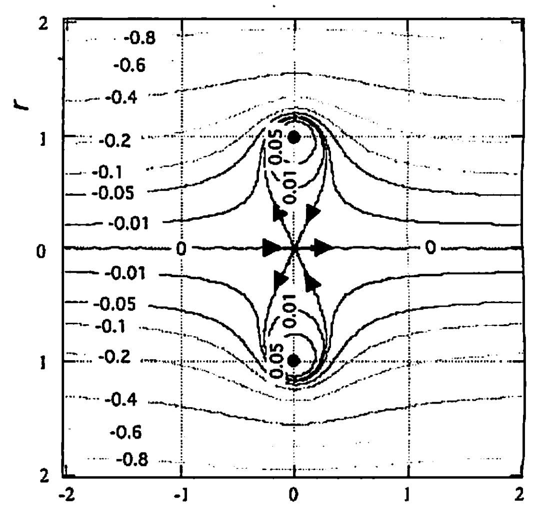

Fig.3 Streamlines around a vortex ring
moving with critical speed
(R=1, Γ=1, U= 0.5)

Figure 4 is the third case of the flow where the uniform flow strength is twice of the critical speed. It is the case of fast moving vortex ring relatively to the space. As described above, the outer flow can go through the inside of the vortex ring. The familiar flow field around the cigarette smoke ring may correspond to this figure.

Comparing Figures 2, 3 and 4, it is cleared that only the relative moving speed to the critical speed, which is the function of the circulation and the radius of the vortex ring, decides the flow fields.

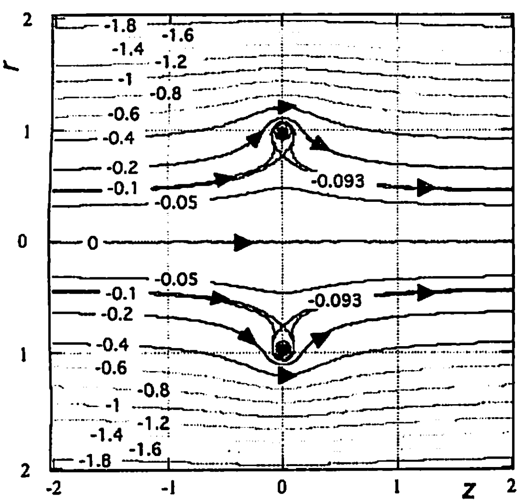

Fig.4 Streamlines around a fast moving vortex ring
(R=1, Γ=1, U= 1)

Professor Batchelor showed the same three flow fields induced by vortex rings in his text book [2], however, they were described as the possibilities of the flow field. He indicated that the differences of the flow pattern among these figures are only come from the character of vortex rings not come from the outer flow strength. He also taught that the outer fluid could go through the inside of vortex ring only when the ratio of strength of vortex filament to ring radius is smaller than a critical value. Theoretically, he analyzed self-induced moving speed of a vortex ring approximately as the following Equation (4), where ε means the small radius of the "tube vortices".

$$U = \frac{\Gamma}{4\pi R} ln(\frac{R}{\varepsilon}) \qquad (4)$$

Actually, Batchelor and any other professors did not discuss the superimposing uniform flows on a vortex ring inducing flow field, so that, they seem not to accept the motion of vortex ring with arbitrary velocity in the fluid filled space at rest. Furthermore, it has been understood for long time that the speed of self-moving vortex ring could not calculated, otherwise be infinite, because of the mathematical singularity of the ring filament.

2.3 Cases of a stationary vortex pair

Equation (5) and Figure 5 describe the flow field induced by a vortex pair, a pair of parallel straight vortex lines. Since the flow velocity far apart the origin becomes zero and the vortex ring is fixed to the space, the vortex pair does not move relatively to the space. In this case, the critical outer uniform flow is obtained as in Equation (6).

$$\psi = -\frac{\Gamma}{2\pi} ln(r_1) + \frac{\Gamma}{2\pi} ln(r_2) \qquad (5)$$

$$= -\frac{\Gamma}{2\pi} ln\left(\frac{r_1}{r_2}\right)$$

$$r_{1,2} = \left[(x \mp X)^2 + (y-Y)^2\right]^{1/2}$$

$$U = 2\frac{\Gamma}{2\pi X} = \frac{\Gamma}{\pi X} = U_{cri} \quad \text{(Critical speed)} \qquad (6)$$

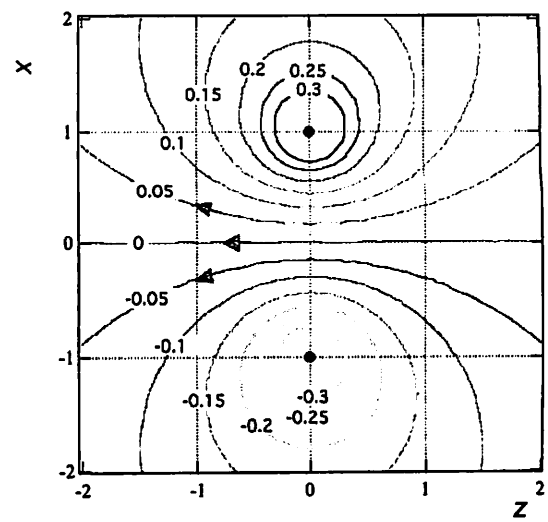

Fig.5 Streamlines around a stationary vortex pair
(X=1, -1, Γ=1, U=0)

2.4 Cases of several moving vortex pairs

Figure 6 shows flow field around a slowly moving vortex pair. It is a special case of the flow when the superimposed uniform flow speed is one fourth of the critical speed. In this case, the flow accompanied with the vortex pair is circulating only in a near circular cylindrical zone. One might call it a circular cylindrical vortex flow like as Hill's vortex sphere.

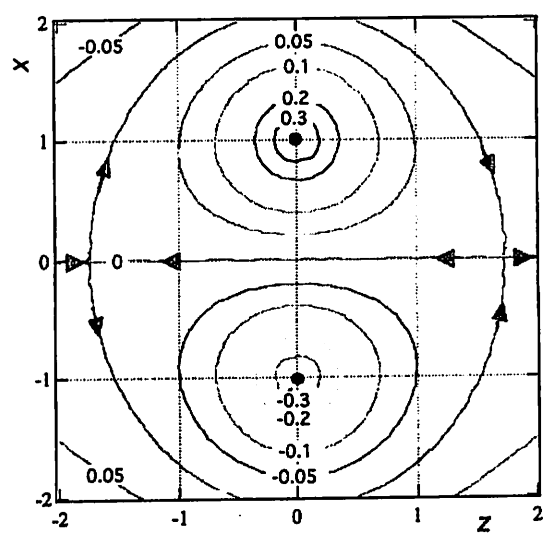

Fig.6 Streamlines around a slowly moving vortex pair
(X=1, -1, Γ=1, U= $U_{cri}/4$)

Figure 7 shows the critical flow field around a moving vortex pair. The outer uniform flow and the vortex inducing flow is cancelled each other at the center of the figure. If the outer flow strength or the moving speed of the vortex pair is equal or under the critical speed, the outer flow never goes through the field area between the vortex lines.

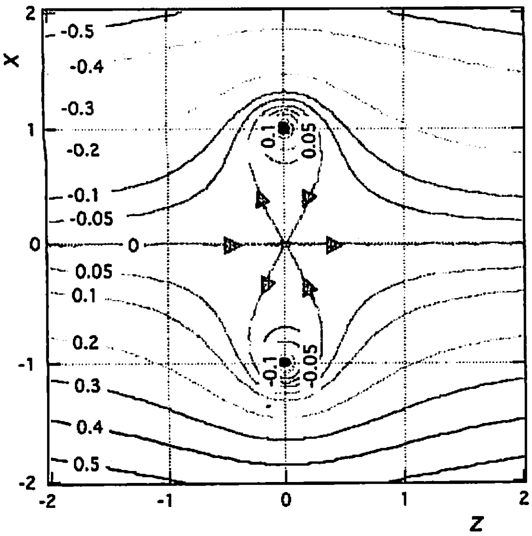

Fig.7 Streamlines around a vortex pair moving with critical speed (X=1, -1, Γ=1, U= $Ucri$)

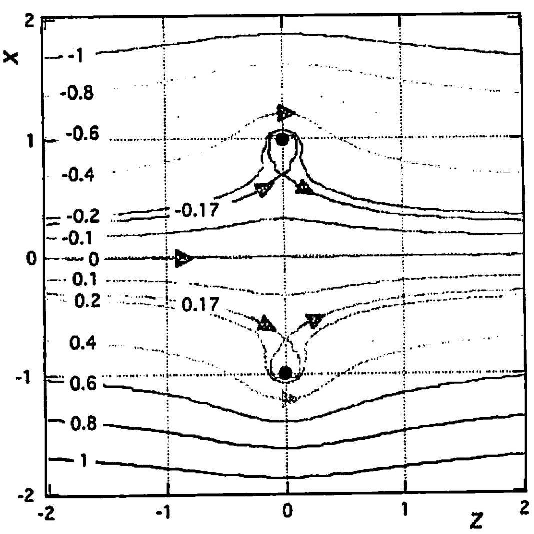

Fig.8 Streamlines around a fast moving vortex pair (X=1, -1, Γ=1, U= 2 $Ucri$)

Figure 8 shows another special case of the flow when the outer uniform flow has the strength twice the critical speed. In this case, the outer flow can go through the inner area between two vortex lines with the center speed just same as the critical speed but opposite flow direction. Obviously, this is the case of a fast moving vortex pair in the fluid filled space at rest.

Professor Prandtl seemed to have another understanding about these flows. In his text book [3], Prandtl told that the difference between Figure 5 and Figure 6 only came from the difference of observation systems and so that two flows were the same. He seems to have same misunderstanding with Batchelor. They might have a conviction that any vortex rings and any vortex pairs should move by their own inducing flow fields.

In Japanese textbook, Professor Tatsumi [4] for example showed the same streamlines of Figure 6 when a vortex pair moves at the speed of one fourth of the authors cold critical speed, because that is the induced speed usually understood. Tatsumi said that the figure is observed by the coordinate fixed to the vortex pair just like as Prandtl told. That is true in the case of slowly moving vortex pairs, however, if the case of stationary vortex pairs, the flow field becomes to that as in Figure 5. It might be declared that these two or four figures are entirely different flow fields to each other.

3. DISCUSSION and CONCLUSSION

It is cleared that any vortex rings or any vortex pairs do not necessarily move by their own inducing flow fields, and that, the flow pattern is decided by the relative strengths of outer uniform flow and vortices, in other words, relative moving speed of vortex rings/pairs to the critical speed. So far, almost all textbooks of fluid dynamics have described as if vortex rings and vortex pairs should move at a fixed speed by their own self or mutual inducing flow fields.

The authors do accept the possibility of movements of a vortex ring and pair. However, they do not agree the concept that every vortex ring and pair should move at only one fixed velocity as so far understood. There might have been a crucial misunderstanding on fluid dynamics during a centurial age. One of the reasons of misunderstanding may come from a simple confusing about the relative observation systems of flow around a moving body. For the proof, it is easy to say that usually we can understand the difference of flow fields between a stationary doublet, i.e., source and sink, and a moving doublet which becomes a flow around a moving circular cylindrical body.

The other reason of misunderstanding might come from the mathematical singularities of vortex filaments. It may be said that superimposing of any flows just only on a point of singularity is meaningless, because the singularity does not be affected by the superimposition. So that, calculated well-known infinitive speed of vortex ring filament and the one fourth of critical speed in the case of vortex pair filaments may be judged simultaneously to be meaningless.

A question may be left, say, in a mass-like problem of vortex ring and vortex lines. In actual physical problems, tornadoes or typhoons for example, two vortices may be interacted by each other, and usually the week vortex shall be influenced much more than the strong vortices. However, how could we decide the mass of vortex ring filament or line filaments? Zero mass and infinitive speed is thus the mathematical singularity. Any vortex does not be affected by any superposition and may move at any arbitrary speed decided by the other physical boundaries or initial conditions. The flow patterns of Figure 1 to Figure 8 are concluded as the different cases of a stationary and/or moving vortex ring and a vortex pair. Any vortex rings and pairs should not necessarily move at the fixed speed that might be misunderstood for long time.

Finally, it must be noticed once again that the above calculations and discussions are restricted to the inviscid steady potential flow theories. It excludes the effect of viscosities of actual fluid, nor the generation, stability and decay processes of a physical vortex ring or pair. As the matter of fact, Professor Bachelor discussed the stability of a laminar vortex ring and studied the vortex ring generation in the wakes behind bluff bodies at large Reynolds number (5). Then, he concluded the possibilities of the stable laminar vortex rings as described in regarding Figures 2, 3 and 4. On the other hand, our discussions and the findings are in the purely inviscid mathematical problems of the ideal flow fields but with singularities of vortex filaments.

Since the topics might be concerned with a crucial/centurial misunderstanding on fluid dynamics, authors hopefully request sooner inspection of the article and then revisions of textbooks and the other vortex connecting papers on fluid dynamics.

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Professor M. Inoue and Professor Emeritus Y. Senoo, Kyushu University, kindly discussed with authors on these somewhat confusing but important agenda and gave them much useful advises. Other Japanese professors in JSME and in the Society of Fluid Mechanics could not understand the authors' points of findings and gave them the fairly negative comments. Professor Max. Platzer, Naval Postgraduate School, has showed much interest and advised authors more prudent inspection to the findings introducing an article on Vortex Rings (6) and a book of Fluid Vortices (7). Platzer also introduced comments of Professor Sarpkaya at the same school, although those were not approval with authors'. Finally, Dr. G. Yates, a colleague of one of the authors at their time of California Institute of Technology, has sent authors fairly positive comments. He said "Unless real fluid effects are included and due consideration is given to the forces required to move (or keep stationary) the vortices, all solutions are possible. The required forces may or may not be feasible to apply".

Professor Ali Ogut, at Rochester Institute of Technology and the Chief Organizer of the 4th ASME/JSME Joint Fluid Engineering Conference, planed and called for collaborations with authors on a panel concerning the moving vortices in the Conference. However, the plan has been postponed to the next chance of the Conference.

Authors express their sincere appreciations to all discussers of these academic fields of fluid dynamics and hope to have another chance to discuss on the problem in near future.

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