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AN ELEMENTARY PROOF OF A THEOREM OF T. F. HAVEL

Yasuhiko KAMIYAMA

1. Introduction

We consider the configuration space of the planar equilateral pentagon linkage. More precisely, we define $M$ by

$$M = \{(z_1, z_2, z_3) \in \mathbb{R}^6; \|z_i - z_{i+1}\| = 1, i = 1, 2, \ldots 5\}$$

where $z_4$ and $z_5$ are fixed vectors in $\mathbb{R}^2$ and we regard $z_6$ as $z_1$.

Note that the freedom of independent parameters of $M$ equals to 2. Then it is natural to ask whether $M$ is a manifold and, if in this case, what kind of manifold. In [2], T. F. Havel answers this question and the result is as follows.

Theorem 1. $M$ is a compact, connected and orientable two-dimensional manifold of genus 4.

In order to prove this theorem, Havel considers the following steps. (1) First prove that $M$ is a smooth manifold by showing local coordinates explicitly. (2) Next make a function $f : M \to \mathbb{R}$ by assigning a point of $M$ to its directed area. Then prove that $f$ is a Morse function and $\chi(M) = -6$ is obtained by the Morse theory, here $\chi(M)$ is the Euler number of $M$. (3) Finally prove that $M$ is orientable.

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It will be natural that one hopes to prove Theorem 1 more directly without using the Morse theory. And the purpose of this paper is to execute this.

2. Geometric proof of Theorem 1

We write the coordinates of $z_4$ and $z_5$ by $z_4 = (-1/2, 0)$ and $z_5 = (1/2, 0)$ respectively and write the clockwise angle from the vector $z_5 z_4$ to $z_5 z_1$ by $\alpha$ and the counterclockwise angle from the vector $z_4 z_5$ to $z_4 z_3$ by $\beta$ respectively. It is clear that $z_1 = (1/2 - \cos \alpha, \sin \alpha), z_3 = (-1/2 + \cos \beta, \sin \beta)$.

Fix $z_1$ and $z_3$ arbitrarily, then the freedom of $z_2$ will be given by the following:

(i) If $0 < ||z_1 - z_3|| < 2$, then we can take $z_2$ at exactly 2 different points. In fact if $z_2$ is taken so that $||z_1 - z_2|| = 1, ||z_2 - z_3|| = 1$, then the symmetric point $z'_2$ of $z_2$ with respect to the segment $z_1 z_3$ also satisfies $||z_1 - z'_2|| = 1, ||z'_2 - z_3|| = 1$.

(ii) If $||z_1 - z_3|| = 2$, then we can take $z_2$ at exactly one point. In fact in this case $z_2$ should be the middle point of the segment $z_1 z_3$.

(iii) If $||z_1 - z_3|| = 0$, then the freedom of $z_2$ is homeomorphic to $S^1$. In fact in this case $z_2$ can be taken at any point of the circle of radius 1 centered at $z_1 = z_3$.

(iv) If $2 < ||z_1 - z_3||$, then it is clear that we cannot take $z_2$ at any point.

Note that the case (iii) occurs if and only if $\alpha = \beta = \pi/3$ or $\alpha = \beta = 5\pi/3$.

Let $R$ be the subspace of $M$ consisting of points of the cases (i)
or (ii) and let \( D \) be the subspace of \( T^2 \) consisting of \((\alpha, \beta)\) such that \( 0 < || z_1 - z_3 || \leq 2 \), where \( T^2 \) is the 2 dimensional torus obtained from \([0, 2\pi] \times [0, 2\pi]\) by the identification \((\alpha, 0) \sim (\alpha, 2\pi)\) and \((0, \beta) \sim (2\pi, \beta)\). Note that the boundary of \( D \), which will be denoted by \( \partial D \), consists of points of the case (ii).

Thus \( R \) will be obtained by the following manner. Let \( D^{(1)} \) and \( D^{(2)} \) be two copies of \( D \), \( \partial D^{(1)} \) and \( \partial D^{(2)} \) be boundary of \( D^{(1)} \) and \( D^{(2)} \) respectively and let \( i : \partial D^{(1)} \to \partial D^{(2)} \) be the identity map. Then \( R \) is homeomorphic to \( D^{(1)} \bigcup D^{(2)} / \sim \), where \( D^{(1)} \bigcup D^{(2)} \) is the disjoint union of \( D^{(1)} \) and \( D^{(2)} \) and the identification \( \sim \) is given by \( z^{(1)} \sim z^{(2)} \) if and only if \( z^{(1)} \in \partial D^{(1)} \), \( z^{(2)} \in \partial D^{(2)} \) such that \( z^{(2)} = i z^{(1)} \).

Because of the above observations, we first investigate the domain \( \bar{D} \) which is defined by \( \bar{D} = \{(\alpha, \beta) \in T^2; || z_1 - z_3 || \leq 2 \} \). In order to do this, we shall see \( \partial \bar{D} \), which is by definition \( \{(\alpha, \beta) \in T^2; || z_1 - z_3 || = 2 \} \).

**Lemma 2.1.** \( \partial \bar{D} \) is homeomorphic to \( S^1 \).

**Proof.** This lemma seems clear from the definition of \( \partial \bar{D} \). But for the completeness we shall give some details.

Note that

\[
(2.2) \quad \partial \bar{D} = \{(\alpha, \beta) \in T^2; (1 - \cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = 4}\.
\]

Once \( \alpha \in [0, 2\pi] \) is fixed, then \( \beta \) will be given by

\[
(2.3) \quad \beta = -\alpha/2 + \sin^{-1}\{(1/2 + \cos \alpha)/(-2 \sin(\alpha/2))\}
\]

where \( \sin^{-1} z = \{y \in (-\infty, \infty); \sin y = z \} \).
Note that (2.3) asserts that \(-1 \leq (1/2 + \cos \alpha)/(\sin(\alpha/2)) \leq 1\). Then we can easily show that \(\alpha\) must satisfy \(\pi/3 \leq \alpha \leq 5\pi/3\) such that if \(\alpha = \pi/3\), then \(\beta = 4\pi/3\) and \(\alpha = 5\pi/3\), then \(\beta = 2\pi/3\).

By using these results, we can easily prove that \(\partial D\) is homeomorphic to \(S^1\). □

By using Lemma 2.1, we can show that \(D\) is homeomorphic to \(T^2 - e^2\), where \(e^2\) is a small open disk contained in \(T^2\). Hence \(D\) is homeomorphic to \(T^2 - \{e^2 \cup p_1 \cup p_2\}\), where \(p_1\) corresponds to \(\alpha = \beta = \pi/3\) and \(p_2\) corresponds to \(\alpha = \beta = 5\pi/3\).

Recall that \(R\) is homeomorphic to \(D^{(1)} \cup D^{(2)}/\sim\). Hence we have the following:

**Proposition 2.4.** \(R\) is homeomorphic to \(\Sigma_2 - \{p_1^{(1)}, p_1^{(2)}, p_2^{(1)}, p_2^{(2)}\}\), where \(\Sigma_2\) is the Riemannian surface of genus 2 and \(\{p_1^{(1)}, p_1^{(2)}\}\) are copies of \(p_1\), \(\{p_2^{(1)}, p_2^{(2)}\}\) copies of \(p_2\).

Next we shall investigate the case of (iii), i.e. the situation around \(p_1^{(1)}, p_1^{(2)}, p_2^{(1)}\) and \(p_2^{(2)}\) in \(R\). We think of a small closed neighborhood of \(p_1^{(1)}\) as \(CS^1 - \{p_1^{(1)}\}\), where \(CS^1\) is the cone of \(S^1\) and the vertex corresponds to \(p_1^{(1)}\). We also consider a small closed neighborhood of \(p_1^{(2)}\) in the same manner. Then by the insights (i) and (iii), it is clear that the topology around \(p_1^{(1)}\) and \(p_1^{(2)}\) is given by the following: First consider \(CS^1 \vee CS^1\) (\(= \) one point union of two \(CS^1\)'s attached by the vertices). Then replace the vertex by \(S^1\).

Note that \(CS^1 \vee CS^1\) changes into \(S^1 \times [0, 1]\) by this operation. Hence we have proved that the topology around \(p_1^{(1)}\) and \(p_1^{(2)}\) is \(S^1 \times [0, 1]\).
If we consider the situation around \( p_2^{(1)} \) and \( p_2^{(2)} \) in the same manner, then we have the following:

**Proposition 2.5.** Let \( M' \) be \( \Sigma_2 \) attached with two \( S^1 \times [0,1]'s \) in some manner. Then \( M \) is homeomorphic to \( M' \).

Finally we prove that \( M \) is orientable. In order to do this, we shall investigate how two \( S^1 \times [0,1]'s \) are attached to \( \Sigma_2 \).

We cut off a small open neighborhood of the vertex in \( CS^1 \) and write the remaining subspace of \( CS^1 \) by \( S^1 \times [0,1-\epsilon] \), where \( \epsilon > 0 \) is small enough. Note that the \( S^1 \times [0,1] \) around \( p_1^{(1)} \) and \( p_1^{(2)} \) is obtained by \( S^1 \times [0,1-\epsilon] \bigcup S^1 \times [0,1-\epsilon]/\sim \), where \( \sim \) is induced by a homeomorphism \( g : S^1 \times \{1-\epsilon\} \to S^1 \times \{1-\epsilon\} \). (c.f. the identification of \( R \) with \( D(1) \bigcup D(2)/\sim \)). We think of \( g \) as \( g : S^1 \to S^1 \). Then \( g \) is given by the following:

**Lemma 2.6.** \( g \) is homotopic to the antipodal map.

Proof. Note that \( CS^1 - \{\text{vertex}\} \) is parametrized by \( \alpha, \beta \), and the freedom \( S^1 \) in the case (iii) is of course parametrized by \( z_2 \). Moreover note that \( z_2 \) and \( z_2' \) corresponds to each other in the case (i). By using these facts, it is easy to see that \( g \) is homotopic to the antipodal map.

\[ \square \]

If we consider the situation around \( p_2^{(1)} \) and \( p_2^{(2)} \) in the same manner, then finally we have the following:

**Theorem 2.7.** Let \( X \) be \( T^2 - \{e_1^2, e_2^2, e_3^2\} \), where \( \{e_1^2, e_2^2, e_3^2\} \) are small open disks. Then \( M' \) is homeomorphic to \( X \bigcup X/\cong \), where \( \cong \) is meant to identify the boundaries of two \( X \)'s via one identity map.
and two maps which are homotopic to the antipodal map.

Note that antipodal map preserves orientation. Hence it is easy to see that $M$ is orientable.

This completes the proof of Theorem 1.

References


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