AN ELEMENTARY PROOF OF A THEOREM OF T. F. HAVEL

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1. Introduction

We consider the configuration space of the planar equilateral pentagon linkage. More precisely, we define $M$ by

$$M = \{(z_1, z_2, z_3) \in \mathbb{R}^6; \| z_i - z_{i+1} \| = 1, i = 1, 2, \ldots, 5\}$$

where $z_4$ and $z_5$ are fixed vectors in $\mathbb{R}^2$ and we regard $z_6$ as $z_1$.

Note that the freedom of independent parameters of $M$ equals to 2. Then it is natural to ask whether $M$ is a manifold and, if in this case, what kind of manifold. In [2], T. F. Havel answers this question and the result is as follows.

Theorem 1. $M$ is a compact, connected and orientable two-dimensional manifold of genus 4.

In order to prove this theorem, Havel considers the following steps. (1) First prove that $M$ is a smooth manifold by showing local coordinates explicitly. (2) Next make a function $f : M \to \mathbb{R}$ by assigning a point of $M$ to its directed area. Then prove that $f$ is a Morse function and $\chi(M) = -6$ is obtained by the Morse theory, here $\chi(M)$ is the Euler number of $M$. (3) Finally prove that $M$ is orientable.

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It will be natural that one hopes to prove Theorem 1 more directly without using the Morse theory. And the purpose of this paper is to execute this.

2. Geometric proof of Theorem 1

We write the coordinates of $z_4$ and $z_5$ by $z_4 = (-1/2, 0)$ and $z_5 = (1/2, 0)$ respectively and write the clockwise angle from the vector $z_5 \vec{z}_4$ to $z_5 \vec{z}_1$ by $\alpha$ and the counterclockwise angle from the vector $z_4 \vec{z}_5$ to $z_4 \vec{z}_3$ by $\beta$ respectively. It is clear that $z_1 = (1/2 - \cos \alpha, \sin \alpha), z_3 = (-1/2 + \cos \beta, \sin \beta)$.

Fix $z_1$ and $z_3$ arbitrarily, then the freedom of $z_2$ will be given by the following:

(i) If $0 < || z_1 - z_3 || < 2$, then we can take $z_2$ at exactly 2 different points. In fact if $z_2$ is taken so that $|| z_1 - z_2 || = 1, || z_2 - z_3 || = 1$, then the symmetric point $z'_2$ of $z_2$ with respect to the segment $z_1z_3$ also satisfies $|| z_1 - z'_2 || = 1, || z'_2 - z_3 || = 1$.

(ii) If $|| z_1 - z_3 || = 2$, then we can take $z_2$ at exactly one point. In fact in this case $z_2$ should be the middle point of the segment $z_1z_3$.

(iii) If $|| z_1 - z_3 || = 0$, then the freedom of $z_2$ is homeomorphic to $S^1$. In fact in this case $z_2$ can be taken at any point of the circle of radius 1 centered at $z_1 = z_3$.

(iv) If $2 < || z_1 - z_3 ||$, then it is clear that we cannot take $z_2$ at any point.

Note that the case (iii) occurs if and only if $\alpha = \beta = \pi/3$ or $\alpha = \beta = 5\pi/3$.

Let $R$ be the subspace of $M$ consisting of points of the cases (i)
or (ii) and let $D$ be the subspace of $T^2$ consisting of $(\alpha, \beta)$ such that $0 < \| z_1 - z_3 \| \leq 2$, where $T^2$ is the 2 dimensional torus obtained from $[0, 2\pi] \times [0, 2\pi]$ by the identification $(\alpha, 0) \sim (\alpha, 2\pi)$ and $(0, \beta) \sim (2\pi, \beta)$. Note that the boundary of $D$, which will be denoted by $\partial D$, consists of points of the case (ii).

Thus $R$ will be obtained by the following manner. Let $D^{(1)}$ and $D^{(2)}$ be two copies of $D$, $\partial D^{(1)}$ and $\partial D^{(2)}$ be boundary of $D^{(1)}$ and $D^{(2)}$ respectively and let $i : \partial D^{(1)} \rightarrow \partial D^{(2)}$ be the identity map. Then $R$ is homeomorphic to $D^{(1)} \bigsqcup D^{(2)} / \sim$, where $D^{(1)} \bigsqcup D^{(2)}$ is the disjoint union of $D^{(1)}$ and $D^{(2)}$ and the identification $\sim$ is given by $z^{(1)} \sim z^{(2)}$ if and only if $z^{(1)} \in \partial D^{(1)}$, $z^{(2)} \in \partial D^{(2)}$ such that $z^{(2)} = iz^{(1)}$.

Because of the above observations, we first investigate the domain $\tilde{D}$ which is defined by $\tilde{D} = \{ (\alpha, \beta) \in T^2; \| z_1 - z_3 \| \leq 2 \}$. In order to do this, we shall see $\partial \tilde{D}$, which is by definition $\{ (\alpha, \beta) \in T^2; \| z_1 - z_3 \| = 2 \}$.

**Lemma 2.1.** $\partial \tilde{D}$ is homeomorphic to $S^1$.

Proof. This lemma seems clear from the definition of $\partial \tilde{D}$. But for the completeness we shall give some details.

Note that

$$
(2.2) \quad \partial \tilde{D} = \{ (\alpha, \beta) \in T^2; (1 - \cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = 4 \}.
$$

Once $\alpha \in [0, 2\pi]$ is fixed, then $\beta$ will be given by

$$
(2.3) \quad \beta = -\alpha/2 + \sin^{-1}\{(1/2 + \cos \alpha)/(-2 \sin(\alpha/2))\}
$$

where $\sin^{-1} \, x = \{ y \in (-\infty, \infty); \sin y = x \}$. 

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Note that (2.3) asserts that \(-1 \leq (1/2 + \cos \alpha)/(−2 \sin(\alpha/2)) \leq 1\). Then we can easily show that \(\alpha\) must satisfy \(\pi/3 \leq \alpha \leq 5\pi/3\) such that if \(\alpha = \pi/3\), then \(\beta = 4\pi/3\) and \(\alpha = 5\pi/3\), then \(\beta = 2\pi/3\).

By using these results, we can easily prove that \(\partial \tilde{D}\) is homeomorphic to \(S^1\). \(\square\)

By using Lemma 2.1, we can show that \(\tilde{D}\) is homeomorphic to \(T^2 - e^2\), where \(e^2\) is a small open disk contained in \(T^2\). Hence \(D\) is homeomorphic to \(T^2 - \{e^2 \cup p_1 \cup p_2\}\), where \(p_1\) corresponds to \(\alpha = \beta = \pi/3\) and \(p_2\) corresponds to \(\alpha = \beta = 5\pi/3\).

Recall that \(R\) is homeomorphic to \(D(1) \bigsqcup D(2)/\sim\). Hence we have the following:

**Proposition 2.4.** \(R\) is homeomorphic to \(\Sigma_2 - \{p_1^{(1)}, p_1^{(2)}, p_2^{(1)}, p_2^{(2)}\}\), where \(\Sigma_2\) is the Riemannian surface of genus 2 and \(\{p_1^{(1)}, p_1^{(2)}\}\) are copies of \(p_1\), \(\{p_2^{(1)}, p_2^{(2)}\}\) copies of \(p_2\).

Next we shall investigate the case of (iii), i.e. the situation around \(p_1^{(1)}, p_1^{(2)}, p_2^{(1)}\) and \(p_2^{(2)}\) in \(R\). We think of a small closed neighborhood of \(p_1^{(1)}\) as \(CS^1 - \{p_1^{(1)}\}\), where \(CS^1\) is the cone of \(S^1\) and the vertex corresponds to \(p_1^{(1)}\). We also consider a small closed neighborhood of \(p_1^{(2)}\) in the same manner. Then by the insights (i) and (iii), it is clear that the topology around \(p_1^{(1)}\) and \(p_1^{(2)}\) is given by the following: First consider \(CS^1 \vee CS^1\) (\(=\) one point union of two \(CS^1\)'s attached by the vertices). Then replace the vertex by \(S^1\).

Note that \(CS^1 \vee CS^1\) changes into \(S^1 \times [0, 1]\) by this operation. Hence we have proved that the topology around \(p_1^{(1)}\) and \(p_1^{(2)}\) is \(S^1 \times [0, 1]\).
If we consider the situation around $p_2^{(1)}$ and $p_2^{(2)}$ in the same manner, then we have the following:

**Proposition 2.5.** Let $M'$ be $\Sigma_2$ attached with two $S^1 \times [0,1]$'s in some manner. Then $M$ is homeomorphic to $M'$.

Finally we prove that $M$ is orientable. In order to do this, we shall investigate how two $S^1 \times [0,1]$'s are attached to $\Sigma_2$.

We cut off a small open neighborhood of the vertex in $CS^1$ and write the remaining subspace of $CS^1$ by $S^1 \times [0,1-\epsilon]$, where $\epsilon > 0$ is small enough. Note that the $S^1 \times [0,1]$ around $p_1^{(1)}$ and $p_1^{(2)}$ is obtained by $S^1 \times [0,1-\epsilon] \bigsqcup S^1 \times [0,1-\epsilon]/\sim$, where $\sim$ is induced by a homeomorphism $g : S^1 \times \{1-\epsilon\} \to S^1 \times \{1-\epsilon\}$. (c.f. the identification of $R$ with $D^{(1)} \bigsqcup D^{(2)}/\sim$). We think of $g$ as $g : S^1 \to S^1$. Then $g$ is given by the following:

**Lemma 2.6.** $g$ is homotopic to the antipodal map.

Proof. Note that $CS^1 - \{\text{vertex}\}$ is parametrized by $\alpha, \beta$, and the freedom $S^1$ in the case (iii) is of course parametrized by $x_2$. Moreover note that $x_2$ and $x'_2$ corresponds to each other in the case (i). By using these facts, it is easy to see that $g$ is homotopic to the antipodal map.

If we consider the situation around $p_2^{(1)}$ and $p_2^{(2)}$ in the same manner, then finally we have the following:

**Theorem 2.7.** Let $X$ be $T^2 - \{e_1^2, e_2^2, e_3^2\}$, where $\{e_1^2, e_2^2, e_3^2\}$ are small open disks. Then $M'$ is homeomorphic to $X \bigsqcup X / \cong$, where $\cong$ is meant to identify the boundaries of two $X$'s via one identity map.
and two maps which are homotopic to the antipodal map.

Note that antipodal map preserves orientation. Hence it is easy to see that $M$ is orientable.

This completes the proof of Theorem 1.

References


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