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CONVERSE RESULTS FOR THE BEST APPROXIMATION IN BANACH SPACES

TOSHIHIKO NISHISHIRAIHO

ABSTRACT. Converse results for the order of magnitude of the degree of the best approximation in Banach spaces are derived by means of the moduli of continuity of higher orders with respect to a strongly continuous group of multiplier operators.

1. Introduction

Let $C_{2\pi}$ denote the Banach space of all $2\pi$-periodic, continuous functions $f$ on the real line $\mathbb{R}$ with the norm

$$
\|f\|_\infty = \max\{|f(t)| : |t| \leq \pi\}.
$$

Let $\mathbb{N}$ be the set of all positive integers, and put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For each $n \in \mathbb{N}_0$, we denote by $\mathcal{T}_n$ the set of all trigonometric polynomials of degree at most $n$. For a given function $f \in C_{2\pi}$, we define

$$
E_n(C_{2\pi}; f) = \inf\{\|f - g\|_\infty : g \in \mathcal{T}_n\},
$$

which is called the best approximation of degree $n$ to $f$ with respect to $\mathcal{T}_n$. Since $\mathcal{T}_n$ is the $2n+1$-dimensional Chebyshev subspace of $C_{2\pi}$, for each $f \in C_{2\pi}$, there exists a unique trigonometric polynomial $g_n \in \mathcal{T}_n$ of the best approximation of $f$ with respect to $\mathcal{T}_n$, i.e., such that

$$
E_n(C_{2\pi}; f) = \|f - g_n\|_\infty
$$

(see, e.g., [4; Chap. 2, Theorem 6]).

The classical Weierstrass approximation theorem simply states that the sequence $\{E_n(C_{2\pi}; f) : n \in \mathbb{N}_0\}$ converges to zero as $n$ tends

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to infinity for every \( f \in C_{2\pi} \). It does not say how fast \( E_n(C_{2\pi}; f) \) approaches zero. In general, the smoother function, the faster \( E_n(C_{2\pi}; f) \) tends to zero. The results that guarantee this event are sometimes called the direct theorems of Jackson-type (cf. [2]). Conversely, the inverse theorems of Bernstein-type assert that if \( E_n(C_{2\pi}; f) \) tends rapidly enough to zero, then \( f \) has certain smoothness properties, which are usually given in terms of its modulus of continuity, Lipschitz classes and differentiability properties.

In connection with these results, the theorem of Stechkin states the following (cf. [4; Chap. 4, Theorem 5]): Let \( f \in C_{2\pi} \) and \( k \in \mathbb{N} \). Then there exists a constant \( K_k > 0 \) depending only on \( k \) such that for all \( \delta > 0 \),

\[
\omega_k(C_{2\pi}; f, \delta) \leq K_k \delta^k \sum_{0 \leq n \leq k^{-1}} (n + 1)^{k-1} E_n(C_{2\pi}; f),
\]

where \( \omega_k(C_{2\pi}; f, \delta) \) denotes the \( k \)-th modulus of continuity of \( f \), i.e.,

\[
\omega_k(C_{2\pi}; f, \delta) = \sup \{ \| \Delta_t^k(f) \|_\infty : |t| \leq \delta \}
\]

and

\[
\Delta_t^k(f)(\cdot) = \sum_{m=0}^k (-1)^{k-m} \binom{k}{m} f(\cdot - mt).
\]

The statement analogous to this result also holds for the Banach space \( L^p_{2\pi} \) consisting of all \( 2\pi \)-periodic, \( p \)-th power Lebesgue integrable functions \( f \) on \( \mathbb{R} \) with the norm

\[
\| f \|_p = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p \, dt \right)^{1/p} \quad (1 \leq p < \infty)
\]

using the \( k \)-th integral modulus of continuity of \( f \), i.e.,

\[
\omega_k(L^p_{2\pi}; f, \delta) = \sup \{ \| \Delta_t^k(f) \|_p : |t| \leq \delta \}
\]

(cf. [14; Theorem 6.1.1]).

The purpose of this paper is to extend these results to arbitrary Banach spaces, and in particular, homogeneous Banach spaces (cf. [3], [6], [12]) which include \( C_{2\pi} \) and \( L^p_{2\pi}, 1 \leq p < \infty \), as special cases. For this aim, we consider the following setting:

Let \( X \) be a Banach space with norm \( \| \cdot \|_X \), and let \( B[X] \) denote the Banach algebra of all bounded linear operators of \( X \) into itself with the usual operator norm \( \| \cdot \|_{B[X]} \). Let \( Z \) denote the set of all integers, and let \( \{ P_j : j \in Z \} \) be a sequence of projection operators in \( B[X] \) satisfying the following conditions:
(P-1) The projections \( P_j, j \in \mathbb{Z} \), are mutually orthogonal, i.e., \( P_j P_n = \delta_{j,n} P_n \) for all \( j, n \in \mathbb{Z} \), where \( \delta_{j,n} \) denotes Kronecker's symbol.

(P-2) \( \{ P_j : j \in \mathbb{Z} \} \) is fundamental, i.e., the linear span of the set \( \bigcup_{j \in \mathbb{Z}} P_j(X) \) is dense in \( X \).

(P-3) \( \{ P_j : j \in \mathbb{Z} \} \) is total, i.e., if \( f \in X \) and \( P_j(f) = 0 \) for all \( j \in \mathbb{Z} \), then \( f = 0 \).

For each \( n \in \mathbb{N}_0 \), let \( M_n \) be the linear span of the set \( \{ P_j(X) : |j| \leq n \} \), which is a closed linear subspace of \( X \). For a given \( f \in X \), we define

\[
E_n(X; f) = \inf \{ \| f - g \|_X : g \in M_n \},
\]

which is called the best approximation of degree \( n \) to \( f \) with respect to \( M_n \). Then we have

\[
E_0(X; f) \geq E_1(X; f) \geq \cdots \geq E_n(X; f) \geq E_{n+1}(X; f) \geq \cdots \geq 0,
\]

and Condition (P-2) implies that for every \( f \in X \),

\[
\lim_{n \to \infty} E_n(X; f) = 0. \tag{2}
\]

In [10] (cf. [8], [9]) and [11], we studied the relation between the rapidity of convergence (2) and certain smoothness properties of \( f \) in terms of its moduli of continuity with respect to a strongly continuous group of multiplier operators on \( X \) associated with Fourier series expansions corresponding to \( \{ P_j : j \in \mathbb{Z} \} \). Here we shall further make use of this our way of thinking.

2. Preliminary results

For any \( f \in X \), we associate its (formal) Fourier series expansion (with respect to \( \{ P_j : j \in \mathbb{Z} \} \))

\[
f \sim \sum_{j=-\infty}^{\infty} P_j(f). \tag{3}
\]

An operator \( T \in B[X] \) is called a multiplier operator on \( X \) if there exists a sequence \( \{ \tau_j : j \in \mathbb{Z} \} \) of scalars such that for every \( f \in X \),

\[
T(f) \sim \sum_{j=-\infty}^{\infty} \tau_j P_j(f),
\]
and the following notation is used:

\[ T \sim \sum_{j=-\infty}^{\infty} \tau_j P_j. \]

Thus, this implies that \( P_j T = \tau_j P_j \) for all \( j \in \mathbb{Z} \) (cf. [1], [6], [7], [15]).

Let \( M[X] \) denote the set of all multiplier operators on \( X \), which is a commutative closed subalgebra of \( B[X] \) containing the identity operator \( I \). Let \( \{T_t : t \in \mathbb{R}\} \) be a family of operators in \( M[X] \) satisfying

\[ A = \sup\{\|T_t\|_{B[X]} : t \in \mathbb{R}\} < \infty \]

and having the expansions

\[ T_t \sim \sum_{j=-\infty}^{\infty} e^{\lambda_j t} P_j \quad (t \in \mathbb{R}), \quad (4) \]

where \( \{\lambda_j : j \in \mathbb{Z}\} \) is a sequence of scalars. Then \( \{T_t : t \in \mathbb{R}\} \) becomes a strongly continuous group of operators in \( B[X] \), and there holds

\[ G(f) \sim \sum_{j=-\infty}^{\infty} \lambda_j P_j(f) \quad (f \in D(G)), \quad (5) \]

where \( G \) is the infinitesimal generator of \( \{T_t : t \in \mathbb{R}\} \) with domain \( D(G) \) ([6; Proposition 2]). Furthermore, we have the following.

**Lemma 1.** Let \( k \in \mathbb{N} \). Then we have:

(a) For all \( g \in X \) and all \( j \in \mathbb{Z} \),

\[ G^k(P_j(g)) = \lambda_j^k P_j(g). \]

(b) For all \( f \in D(G^k) \),

\[ G^k(f) \sim \sum_{j=-\infty}^{\infty} \lambda_j^k P_j(f). \]

**Proof.** In view of (4) and (5), this follows from induction on \( k \).

For each \( k \in \mathbb{N}_0 \) and \( t \in \mathbb{R} \), we define

\[ \Delta_t^0 = I, \quad \Delta_t^k = (T_t - I)^k = \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} T_{mt} \quad (k \geq 1), \]

which stands for the \( k \)-th iteration of \( T_t - I \). Then \( \Delta_t^k \) belongs to \( M[X] \), and

\[ \|\Delta_t^k\|_{B[X]} \leq A_k, \]

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where
\[ A_k = \min\{ (A+1)^k, 2^k A \}, \]
and there holds
\[ \Delta_t^k \sim \sum_{j=-\infty}^{\infty} (e^{\lambda_j t} - 1)^k P_j. \]
If \( k \in \mathbb{N}_0, f \in X \) and \( \delta \geq 0 \), then we define
\[ \omega_k(X; f, \delta) = \sup \{ \| \Delta_t^k(f) \|_X : |t| \leq \delta \}, \]
which is called the \( k \)-th modulus of continuity of \( f \) with respect to \( \{ T_t : t \in \mathbb{R} \} \). This quantity has the following properties.

**Lemma 2.** ([10: Lemma 1]) Let \( k \in \mathbb{N} \) and \( f \in X \). Then we have:

(a) For all \( \delta \geq 0 \),
\[ \omega_k(X; f, \delta) \leq A_k \| f \|_X. \]

(b) \( \omega_k(X; f, \cdot) \) is a monotone increasing function on \([0, \infty)\) and \( \omega_k(X; f, 0) = 0 \).

(c) For all \( r \in \mathbb{N}_0 \) and all \( \delta \geq 0 \),
\[ \omega_{k+r}(X; f, \delta) \leq A_k \omega_r(X; f, \delta). \]
In particular, we have
\[ \lim_{\delta \to 0} \omega_k(X; f, \delta) = 0. \]

(d) For all \( \xi, \delta \geq 0 \),
\[ \omega_k(X; f, \xi \delta) \leq A (1 + \xi)^k \omega_k(X; f, \delta). \]

(e) If \( 0 < \delta \leq \xi \), then
\[ \omega_k(X; f, \xi) / \xi^k \leq 2^k A \omega_k(X; f, \delta) / \delta^k. \]

(f) If \( f \in D(G^k) \), then
\[ \omega_{k+r}(X; f, \delta) \leq A \delta^k \omega_r(X; G^k(f), \delta) \]
for all \( r \in \mathbb{N}_0 \) and all \( \delta \geq 0 \).

(g) \( \omega_k(X; \cdot, \delta) \) is a seminorm on \( X \) for each \( \delta \geq 0 \).
Let $k \in \mathbb{N}$ and $\alpha > 0$. An element $f \in X$ is said to satisfy the $k$-th Lipschitz condition of order $\alpha$ with constant $M, M > 0$, or to belong to the class $\text{Lip}_k(X; \alpha, M)$ if $\omega_k(X; f, \delta) \leq M \delta^\alpha$ for all $\delta \geq 0$. $\text{Lip}_k(X; \alpha)$ denotes the class consisting of all $f \in \text{Lip}_k(X; \alpha, M)$ for some constant $M > 0$. In particular, $Z(X) = \text{Lip}_2(X; 1)$ is called the Zygmund class. $\text{lip}_k(X; \alpha)$ denotes the class consisting of all $f \in X$ such that

$$\omega_k(X; f, \delta) = o(\delta^\alpha) \quad (\delta \to +0).$$

Note that

$$D(G^k) \subseteq \text{Lip}_k(X; k) \quad (k \in \mathbb{N})$$

and

$$\text{Lip}_k(X; \alpha) = \text{lip}_k(X; k) \quad (\alpha > k).$$

The following result gives the direct estimate for $E_n(X; f)$ in the Lipschitz classes.

**Lemma 3.** ([10; Corollary 1 (a)]) If $f \in \text{Lip}_k(X; \alpha, M)$ for some $k \in \mathbb{N}$, then for all $n \in \mathbb{N}$,

$$E_n(X; f) \leq AM L_k \frac{1}{n^\alpha},$$

where $L_k$ is a positive constant depending only on $k$.

The following Bernstein-type inequality plays an important role in the derivation of certain smoothness properties of an element $f \in X$ from the hypothesis that the sequence $\{E_n(X; f) : n \in \mathbb{N}_0\}$ tends to zero with a given rapidity.

**Lemma 4.** ([11; Lemma 5]) Let $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$. If $\lambda_j = -ij$ for all $j \in \mathbb{Z}$, then for every $f \in M_n$,

$$\|G^k(f)\|_X \leq (2nB)^k\|f\|_X,$$

where

$$B = \sup\{\|T_i\|_{B[X]} : |t| \leq \pi\}.$$

For any $\lambda \in \mathbb{R}$, $[\lambda]$ denotes the largest integer not exceeding $\lambda$.

**Lemma 5.** Let $\varphi$ be a non-negative, monotone decreasing function defined on $\mathbb{N}$. Let $\{\alpha_j : j \in \mathbb{N}\}$ be a strictly monotone increasing sequence of positive integers such that there exist $\alpha > 1$ and $\beta > 0$ satisfying

$$\frac{\alpha_j}{\alpha_{j-1}} \geq \alpha \quad (j = 0, 1, 2, \ldots) \quad (6)$$
and
\[ \frac{\alpha_j - 1}{\alpha_{j+1}} \geq \beta \quad (j = 0, 1, 2, \cdots), \]  
where \( 0 < \alpha_{-1} < 1 \). Then for all \( m, \ell \in \mathbb{N}_0, m \geq \ell \geq 0 \) and all \( p \in \mathbb{R}, p \geq 1 \),
\[ \sum_{j=\ell}^{m} \alpha_{j+1}^p \varphi(\alpha_j) \leq \frac{1}{\beta^p(\alpha - 1)} \sum_{n=|\alpha_{\ell - 1}| + 1}^{\alpha_m} n^{p-1} \varphi(n). \]  

Proof. By (6) and (7), we have
\[ \sum_{n=|\alpha_{\ell - 1}| + 1}^{\alpha_j} n^{p-1} \varphi(n) \geq \sum_{n=|\alpha_{\ell - 1}| + 1}^{\alpha_j} \alpha_{j-1}^{p-1} \varphi(\alpha_j) \]
\[ \geq (\alpha_j - \alpha_{j-1}) \alpha_{j-1}^{p-1} \varphi(\alpha_j) = \alpha_{j-1} \left( \frac{\alpha_j}{\alpha_{j-1}} - 1 \right) \alpha_{j-1}^{p-1} \varphi(\alpha_j) \]
\[ \geq (\alpha - 1) \beta^p \alpha_{j+1}^p \varphi(\alpha_j). \]

Therefore, there holds
\[ (\alpha - 1) \beta^p \sum_{j=\ell}^{m} \alpha_{j+1}^p \varphi(\alpha_j) \leq \sum_{j=\ell}^{m} \sum_{n=|\alpha_{\ell - 1}| + 1}^{\alpha_j} n^{p-1} \varphi(n) \]
\[ = \sum_{n=|\alpha_{\ell - 1}| + 1}^{\alpha_m} n^{p-1} \varphi(n), \]
which implies (8).

3. Main results

In this section, we suppose that for each \( f \in X \), there exists an element \( f_n \in M_n \) of the best approximation of \( f \) with respect to \( M_n \), i.e., such that
\[ E_n(X; f) = \| f - f_n \|_X \quad (n \in \mathbb{N}_0) \]
and that \( \lambda_j = -ij \) for all \( j \in \mathbb{Z} \).

Remark. If the dimension of \( M_n \) is finite, then every \( f \in X \) has an element of the best approximation with respect to \( M_n \). In particular, if \( \{ h_j, h_j^* \}_{j \in \mathbb{Z}} \) is a fundamental, total, biorthogonal system, then (3) reads
\[ f \sim \sum_{j=-\infty}^{\infty} h_j^*(f) h_j, \]
(cf. [10; Remark 1]) and if \( M_n \) is the linear span of the set \( \{ h_j : |j| \leq n \} \), then for every \( f \in X \) there exists an element of the best approximation of \( f \) with respect to \( M_n \). Also, if \( X \) is a uniformly convex Banach space, then every \( f \in X \) has a unique element of the best approximation with respect to \( M_n \). In particular, if \( X \) is a Hilbert space, then for every \( f \in X \) there exists a unique element of the best approximation of \( f \) with respect to \( M_n \). For the general theory of the best approximation theory in normed linear spaces, we refer to [13].

**Theorem 1.** Let \( a, b, k \in \mathbb{N}, a \geq 2, a > b, \delta \in \mathbb{R}, \delta > 0 \) and \( f \in X \). If \( \delta > b \), then

\[
\omega_k(X; f, \delta) \leq A_k E_0(X; f) \leq \frac{A_k}{b^k} \delta^k E_0(X; f).
\]

If \( \delta \leq b \), then

\[
\omega_k(X; f, \delta) \leq C_k \delta^k \left( E_0(X; f) + \sum_{n=1}^{[b/\delta]} n^{k-1} E_n(X; f) \right),
\]

where

\[
C_k = C_k(a, b, A, B) = \max \left\{ \frac{A_k}{b^k}, \; 2^{k+1} AB^k \right\} \max \left\{ b^k, \; \frac{a^{2k}}{a - 1} \right\}.
\]

Thus, there holds

\[
\omega_k(X; f, \delta) \leq C_k \delta^k \sum_{n=0}^{[b/\delta]} (n + 1)^{k-1} E_n(X; f) \quad (\delta > 0).
\]

**Proof.** Let \( f_n \) be an element of the best approximation of \( f \) with respect to \( M_n \). Suppose that \( \delta > b \). There exists \( g \in X \) such that \( f_0 = P_0(g) \). Therefore, we have

\[
\Delta_k^m(f_0) = \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} T_{mT}(P_0(g))
\]

\[
= \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} e^{\lambda_m t} P_0(g) = 0,
\]

and so Lemma 2 (a) yields

\[
\omega_k(X; f, \delta) = \omega_k(X; f - f_0) \leq A_k \| f - f_0 \|_X,
\]

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which gives (9), since $1 < \delta^k/h^k$. Next we assume that $0 < \delta \leq h$. Let $m \in \mathbb{N}_0$. We put

$$
\alpha_j = a^j b \quad (j \in \mathbb{N}_0), \quad \alpha_{-1} = \frac{b}{a}
$$

and

$$
\varphi(n) = E_n(X; f) \quad (n \in \mathbb{N}), \quad \varphi(\alpha_{-1}) = E_0(X; f), \quad f_{\alpha_{-1}} = f_0.
$$

Then Lemma 5 yields that

$$
\sum_{j=1}^{m+1} \alpha_j^k \varphi(\alpha_{j-1}) \leq \frac{a^{2k}}{a-1} \sum_{n=1}^{\alpha_m} n^{k-1} \varphi(n). \quad (11)
$$

Also, we have, by Lemma 2 (a), (f) and (g),

$$
\omega_k(X; f, \delta) \leq \omega_k(X; f - f_{\alpha_m}, \delta) + \omega_k(X; f_{\alpha_m}, \delta) 
\leq A_k \|f - f_{\alpha_m}\|_X + A \delta^k \|G^k(f_{\alpha_m})\|_X. \quad (12)
$$

Since, by Lemma 1 (a),

$$
G^k(f_{\alpha_{-1}}) = \lambda_0^k f_0 = 0,
$$

we have

$$
G^k(f_{\alpha_m}) = \sum_{j=0}^{m} G^k(f_{\alpha_j} - f_{\alpha_{j-1}}).
$$

Hence, by Lemma 4, we have

$$
\|G^k(f_{\alpha_m})\|_X \leq \sum_{j=0}^{m} \|G^k(f_{\alpha_j} - f_{\alpha_{j-1}})\|_X \leq (2B)^k \sum_{j=0}^{m} \alpha_j^k \|f_{\alpha_j} - f_{\alpha_{j-1}}\|_X,
$$

and

$$
\|f_{\alpha_j} - f_{\alpha_{j-1}}\|_X \leq \|f_{\alpha_j} - f\|_X + \|f - f_{\alpha_{j-1}}\|_X = \varphi(\alpha_j) + \varphi(\alpha_{j-1}) \leq 2\varphi(\alpha_{j-1}) \quad (j \in \mathbb{N}_0).
$$

Consequently, we obtain

$$
\|G^k(f_{\alpha_m})\|_X \leq 2(2B)^k \sum_{j=0}^{m} \alpha_j^k \varphi(\alpha_{j-1}). \quad (13)
$$

We choose $m \in \mathbb{N}_0$ such that

$$
a^m \leq \frac{1}{\delta} < a^{m+1}.
$$

Then we have

$$
A_k \|f - f_{\alpha_m}\|_X \leq A_k \varphi(\alpha_m) a^{k(m+1)} \delta^k = \frac{A_k}{b^k} \delta^k \alpha_{m+1} \varphi(\alpha_m). \quad (14)
$$
Therefore, by (12), (13) and (14) we get

\[ \omega_k(X; f, \delta) \leq B_k \delta^k \sum_{j=0}^{m+1} \alpha_j \varphi(\alpha_{j-1}), \quad (15) \]

where

\[ B_k = \max \left\{ \frac{A_k}{b^k}, 2^{k+1} AB^k \right\}. \]

Now, we have, by (11),

\[
\sum_{j=0}^{m+1} \alpha_j \varphi(\alpha_{j-1}) \leq b^k \varphi(\alpha_{-1}) + \frac{a^{2k}}{a-1} \sum_{n=1}^{m} n^{k-1} \varphi(n) \\
\leq b^k \varphi(\alpha_{-1}) + \frac{a^{2k}}{a-1} \sum_{n=1}^{m} n^{k-1} \varphi(n) \\
\leq \max \left\{ b^k, \frac{a^{2k}}{a-1} \right\} \left( \varphi(\alpha_{-1}) + \sum_{n=1}^{\lfloor b/\delta \rfloor} n^{k-1} \varphi(n) \right),
\]

which together with (15) establishes the desired inequality (10).

**Corollary 1.** Let \( a, b, k \in \mathbb{N}, a \geq 2, a > b \) and \( f \in X \). Then the following statements hold:

(a) If

\[ e_k(X; f) = \sum_{n=1}^{\infty} n^{k-1} E_n(X; f) < \infty, \]

then

\[ \omega_k(X; f, \delta) \leq C_k \left( E_0(X; f) + e_k(X; f) \right) \delta^k \]

for all \( \delta > 0 \).

(b) Let \( \alpha > 0 \) and suppose that there exists a constant \( C > 0 \) such that for all \( n \geq 1 \),

\[ E_n(X; f) \leq C \frac{1}{n^\alpha}. \]

If \( \alpha > k \), then

\[ \omega_k(X; f, \delta) \leq C_k \left( E_0(X; f) + E_1(X; f) + \frac{C}{\alpha - k} \right) \delta^k \]

for all \( \delta > 0 \).

If \( \alpha < k \), then

\[ \omega_k(X; f, \delta) \leq C_k \left( E_0(X; f) \delta^{k-\alpha} + \frac{C(2b)^{k-\alpha}}{k - \alpha} \right) \delta^\alpha \]
for all \( \delta > 0 \).

If \( \alpha = k \), then

\[
\omega_k(X; f, \delta) \leq C_k \left( E_0(X; f) + C(2 + \log b) \right) \delta^k |\log \delta|
\]

for all \( \delta \) with \( 0 < \delta \leq e^{-1} \).

For any \( k \in \mathbb{N} \), we define

\[
W(X; k) = \left\{ f \in X : \omega_k(X; f, \delta) = O(\delta^k |\log \delta|) \quad (\delta \to +0) \right\}.
\]

Note that

\[
W(X; k) \subseteq \text{lip}_k(X; \alpha) \quad (0 < \alpha < k),
\]

and by Lemma 3 and Corollary 1, we obtain the following.

**Corollary 2.** (cf. [11; Theorem 4, Corollary 9 and Theorem 5 (b)]).

(a) Let \( k \in \mathbb{N} \) and \( 0 < \alpha < k \). Then we have

\[
\text{lip}_k(X; \alpha) = \left\{ f \in X : E_n(X; f) = O \left( \frac{1}{n^\alpha} \right) \quad (n \to \infty) \right\}.
\]

In particular, for \( 0 < \alpha < 1 \),

\[
\text{lip}_1(X; \alpha) = \left\{ f \in X : E_n(X; f) = O \left( \frac{1}{n^\alpha} \right) \quad (n \to \infty) \right\}
\]

and for \( 0 < \alpha < 2 \),

\[
\text{lip}_2(X; \alpha) = \left\{ f \in X : E_n(X; f) = O \left( \frac{1}{n^\alpha} \right) \quad (n \to \infty) \right\}.
\]

Thus,

\[
Z(X) = \left\{ f \in X : E_n(X; f) = O \left( \frac{1}{n} \right) \quad (n \to \infty) \right\}.
\]

(b) For all \( k, r \in \mathbb{N} \), we have

\[
\text{lip}_k(X; r) \subseteq W(X; r).
\]

In particular,

\[
Z(X) \subseteq W(X; 1) \subseteq \text{lip}_1(X; \alpha) \quad (0 < \alpha < 1).
\]

Finally, we restrict ourselves to the case where \( X \) is a homogeneous Banach space, i.e., \( X \) satisfies the following conditions:

(H-1) \( X \) is a linear subspace of \( L^1_{2\pi} \) and it is a Banach space with norm \( \| \cdot \|_X \).
(H-2) $X$ is continuously embedded in $L^1_{2\pi}$, i.e., there exists a constant $K > 0$ such that

$$\|f\|_1 \leq K\|f\|_X$$

for all $f \in X$.

(H-3) The left translation operator $T_t$ defined by

$$T_t(f)(\cdot) = f(\cdot - t) \quad (f \in X),$$

is isometric on $X$ for each $t \in \mathbb{R}$.

(H-4) For each $f \in X$, the mapping $t \mapsto T_t(f)$ is strongly continuous on $\mathbb{R}$.

Typical examples of homogeneous Banach spaces are $C_{2\pi}$ and $L^p_{2\pi}$, $1 \leq p < \infty$. For other examples, see [6] (cf. [3], [12]).

Now, we define the sequence $\{P_j : j \in \mathbb{Z}\}$ of projection operators in $B[X]$ by

$$P_j(f)(\cdot) = \hat{f}(j)e^{ij} \quad (f \in X),$$

which satisfies Conditions (P-1), (P-2) and (P-3) just as in Section 1 (cf. [3], [6]). Then (3) reduces to

$$f(\cdot) \sim \sum_{j=-\infty}^{\infty} \hat{f}(j)e^{ij} \quad (f \in X)$$

(cf. Remark) and we have (1) for all $k \in \mathbb{N}, t \in \mathbb{R}$ and all $f \in X$. Also, there holds

$$\Delta_k^t(f)(\cdot) \sim \sum_{j=-\infty}^{\infty} (e^{-ijt} - 1)^k \hat{f}(j)e^{ij} \quad (f \in X).$$

Consequently, all the results obtained in this paper hold under the above setting and in particular, for $k = 1, 0 < \alpha < 1$, Corollary 1 (b) implies [12; Theorem 9.4.5.1]. Furthermore, the last statement of Corollary 2 (a) establishes the theorem of Zygmund type (cf. [16; Theorems 8 and 8'], [5; Chap.V.§4, Theorem 1] in homogeneous Banach spaces.
References


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