<table>
<thead>
<tr>
<th>Title</th>
<th>Birational maps of standard projective plane bundles over algebraic surfaces</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Maeda, Takashi</td>
</tr>
<tr>
<td>Citation</td>
<td>Ryukyu mathematical journal, 9: 5-35</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1996-12-27</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/20.500.12000/16088">http://hdl.handle.net/20.500.12000/16088</a></td>
</tr>
<tr>
<td>Rights</td>
<td></td>
</tr>
</tbody>
</table>
BIRATIONAL MAPS OF
STANDARD PROJECTIVE PLANE BUNDLES
OVER ALGEBRAIC SURFACES

TAKASHI MAEDA

in memory of Akira Takaku

ABSTRACT. Let $X$ be a smooth algebraic surface with the function field $K$ and let $\tau : V \rightarrow X$ be a standard $\mathbb{P}^2$-bundle over $X$, i.e. $\tau$ is a flat contraction morphism of an extremal ray of a smooth projective variety $V$ with the generic fibre isomorphic to a $K$-form of $\mathbb{P}^2$, i.e. $V \times_X \bar{K} = \mathbb{P}^2$ for the algebraic closure $\bar{K}$ of $K$. In this paper, some birational maps from $V$ to a standard $\mathbb{P}^2$-bundle $W$ are represented by compositions of elementary birational morphisms, where $W$ is a standard $\mathbb{P}^2$-bundle over the blow-up of $X$ at a point of the non-smooth locus $\Delta$ of $\tau$. Let $C$ be a smooth curve on $X$ intersecting $\Delta$ transversely at one point. A birational map from $V$ to a standard $\mathbb{P}^2$-bundle over $X$ which is isomorphic over $X - C$, is decomposed into elementary birational morphisms. These are generalizations of the results about standard conic bundles by V. G. Sarkisov (Math. USSR. Izv. 20).

The purpose of this paper is to decompose three types of birational maps of standard $\mathbb{P}^2$-bundles over smooth algebraic surfaces into elementary birational morphisms. Let $K$ be a function field of an algebraic surface defined over an algebraically closed field $k$ of characteristic not equal to 3 and let $V_K$ be a $K$-form of $\mathbb{P}^2$, i.e. $V \times_K \bar{K} \cong \mathbb{P}^2$ for the algebraic closure $\bar{K}$ of $K$. Then it is constructed from $V_K$ a standard $\mathbb{P}^2$-bundle

$$\tau : V \rightarrow X,$$

(cf. [Ma]) i.e. $V$ and $X$ are smooth projective varieties and $\tau$ is a flat contraction morphism of an extremal ray with the generic fibre

Received November 30, 1996
isomorphic to the given $K$-form $V_K \to \text{Spec}(K)$. The non-smooth locus $\Delta$ of (1) is a simple normal crossing curve of $X$ and the geometric fibre over a smooth point of $\Delta$ consists of three components $H_i$ ($i = 1, 2, 3$) with $H_i \cong \mathbb{F}_1$ (one point blow up of $\mathbb{P}^2$), $H_i \cap H_{i+1}$ (resp. $H_i \cap H_{i-1}$) is a fibre (resp. the $(-1)$-curve) on $H_i \cong \mathbb{F}_1$ (where the suffix means mod 3) and $H_1 \cap H_2 \cap H_3$ is a one point. The geometric fibre over a singular point of $\Delta$ is non-reduced with the reduced part isomorphic to the cone over a rational twisted cubic in $\mathbb{P}^3$.

**Theorem.** (I) Let $Y \to X$ be the blow-up at a singular point of $\Delta$. A birational map from the standard $\mathbb{P}^2$-bundle $V$ of (1) to a standard $\mathbb{P}^2$-bundle $W$ over $Y$ is factored by elementary birational morphisms

$$V \leftarrow V_1 \leftarrow V_2 \leftarrow V_3 \rightarrow V' \rightarrow V_4 \rightarrow V_5 \rightarrow W,$$

where $V_3 \rightarrow V'$ (resp. $V' \rightarrow V_4$) is a flop with the exceptional sets $F_3 \to \mathbb{P}^1 \leftarrow F_0$ (resp. $F_2 \to \mathbb{P}^1 \leftarrow F_5$). There are isomorphisms of the conormal bundles

$$C_{F_3/V_3} \cong \mathcal{O}_{F_3}(s + 2f) \oplus \mathcal{O}_{F_3}(s + 2f),$$

$$C_{F_0/V'} \cong \mathcal{O}_{F_0}(s + 2f) \oplus \mathcal{O}_{F_0}(s - f),$$

$$C_{F_2/V'} \cong \mathcal{O}_{F_2}(s + 3f) \oplus \mathcal{O}_{F_2}(s - 2f),$$

$$C_{F_5/V_4} \cong \mathcal{O}_{F_5}(s + 3f) \oplus \mathcal{O}_{F_5}(s + f),$$

with the negative section $s$ and a fibre $f$ of the rational ruled surface $\mathbb{F}_n$ of degree $n$.

(II) Let $Y \to X$ be a blow-up at a smooth point of $\Delta$. A birational map from the standard $\mathbb{P}^2$-bundle $V$ of (1) to a standard $\mathbb{P}^2$-bundle $W$ over $Y$ is factored by

$$V \leftarrow V_1 \leftarrow V_3 \rightarrow W,$$

where $V_1 \rightarrow V_3$ (resp. $V_3 \rightarrow W$) is a flop (resp. a single blow up-and-down) with the exceptional sets $F_1 \to \mathbb{P}^1 \leftarrow F_1$, (resp. $\mathbb{P}^2 \leftarrow \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$).

(III) Let $C$ be a smooth curve on $X$ intersecting transversely at one smooth point of $\Delta$. There is a birational map from the standard $\mathbb{P}^2$-bundle $V$ of (1) to a standard $\mathbb{P}^2$-bundle $W$ over $X$, which is factored by

$$V \leftarrow V_1 \leftarrow V_3 \leftarrow V_5 \rightarrow W.$$
where \( V_1 \leftrightarrow V_3 \) (resp. \( V_3 \rightarrow V_5 \)) is a single blow up-and-down (resp. a flop) with the exceptional sets \( \mathbb{P}^1 \leftarrow \mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2 \) (resp. \( F_0 \rightarrow \mathbb{P}^1 \leftarrow F_1 \)). If \( C \) is disjoint from \( \Delta \) and \( V = \mathbb{P}[E] \) over a neighbourhood of \( C \) with a rank three vector bundle \( E \), then the birational map (III) is nothing but an elementary transformation of the vector bundle \( E \) with center over \( C \).

The decompositions of birational maps of standard conic bundles corresponding to (I), (II), (III) above are appeared in [Sa,p368] (cf. (4.3)). The statements (I), (II), (III) in Theorem are proved in §1, §2, §3, respectively.

Throughout this paper, \( \mathbb{F}_n \) is the rational ruled surface of degree \( n \) with a fibre \( f \) and the \((-n)\)-curve \( s \). The direct sum of line bundles is denoted by \( \mathcal{O}_{\mathbb{F}_n}(s + f, s + 2f) = \mathcal{O}_{\mathbb{F}_n}(s + f) \oplus \mathcal{O}_{\mathbb{F}_n}(s + 2f) \), \( \mathcal{O}(1, 1) = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \) etc. The conormal bundle of a subvariety \( S \) in \( V \) is denoted by \( C_{S/V} \).

0. Preliminaries

(0.1) We recall the construction the standard \( \mathbb{P}^2 \)-bundle over \( X \) from the \( K \)-form \( V_K \) (cf.[Ma]). Let \( A \) be the central simple algebra of rank 9 over \( K \) corresponding to \( V_K \) which is represented by an element of the one-dimensional Galois cohomology set \( H^1(K, PGL_3) \). There are a smooth projective surface \( X \) with the function field isomorphic to \( K \) and a maximal \( \mathcal{O}_X \)-order \( \Lambda \) in \( A \) such that the discriminant curve \( \Delta = \Delta(A, X) \) of \( A \) in \( X \) [A-M,p84] is a simple normal crossing curve on \( X \), and \( \Lambda \otimes R \) (where \( R = \mathcal{O}_{X,p} \) is the local ring of \( X \) at a point \( p \) of \( X \)), is isomorphic to

\[
\begin{align*}
(0.1.1) & \quad (\epsilon, \eta)_{3,R} \quad \text{if} \ p \in X - \Delta, \\
(0.1.2) & \quad (\epsilon, g)_{3,R} \quad \text{if} \ p \in \Delta - \text{Sing}(\Delta), \\
(0.1.3) & \quad (f, g)_{3,R} \quad \text{if} \ p \in \text{Sing}(\Delta).
\end{align*}
\]

Here \( \text{Sing}(\Delta) \) is the singular locus of \( \Delta \), \( \{\epsilon, \eta\} \) are units of \( R \) and \( g = 0 \) (resp. \( fg = 0 \)) is a defining equation of \( \Delta \) at \( p \) in (0.1.2) (resp. (0.1.3)), \( (\epsilon, \eta)_{3,R} \) is the \( R \)-algebra generated by two elements \( x, y \) with relations \( x^3 = \epsilon, y^3 = \eta, yx = \omega xy \) (where \( \omega \) is a cube of unity). The standard \( \mathbb{P}^2 \)-bundle \( V \) over \( X \) associated to \( V_K \) is constructed by gluing standard \( \mathbb{P}^2 \)-bundles \( V_R \) over the local rings
$R = \mathcal{O}_{X,p}$ at each point $p \in X$, which are the intersection of $\mathbb{P}[E_R^\vee]$ and the grassmannian $G_3[\Lambda^\vee \otimes R]$ of 3-quotients of $\Lambda^\vee \otimes R$. Here $E_R$ is the $(\Lambda \otimes R)^*$-subspace of $\wedge^3 \Lambda \otimes R$ (where $(\Lambda \otimes R)^*$ is the unit group of $\Lambda \otimes R$) with $E_R \otimes \tilde{K}$ isomorphic to the third symmetric tensor representation space $H^0(\mathcal{O}_{\mathbb{P}^2}(3))$ of $(\Lambda \otimes \tilde{K})^* \cong GL_3(\tilde{K})$ for an algebraic closure $\tilde{K}$ of $K$, and $E_R^\vee$ and $\Lambda^\vee \otimes R$ are the $R$-duals of $E_R$ and $\Lambda \otimes R$, respectively.

(0.2) The following Lemma is used to describe the flop appearing in the birational maps (I)-(III) in Theorem.

**Lemma.** Let $S \cong \mathbb{F}_n$ be a subvariety of a smooth four-fold $V$ with the conormal bundle $\mathcal{C}_{S/V}$ of $S$ in $V$. Assume $\mathcal{C}_{S/V}|_f \cong \mathcal{O}(1,1)$ for any fibre $f$ of $S \cong \mathbb{F}_n$, and $S \subset V$ is flopped to $S^+ \cong \mathbb{F}_m \subset V^+$. Then

(i) For an integer $a \in \mathbb{Z}$,

$$\mathcal{C}_{S/V} \cong \mathcal{O}_{\mathbb{F}_n}(s + af, s + (a - m)f),$$

$$\mathcal{C}_{S^+/V^+} \cong \mathcal{O}_{\mathbb{F}_m}(s + af, s + (a - n)f).$$

(ii) Assume there is a smooth divisor $D$ of $V$ containing $S$ with the birational transform $D^+$ in $V^+$. Let $C^+ = D^+ \cap S^+$. If $\mathcal{C}_{S/D} \cong \mathcal{O}_{\mathbb{F}_n}(s + (a + b)f)$, then $(C^+)^2 = m + 2b$.

(iii) Assume there is a smooth divisor $F$ of $V$ such that $C = F \cap S$ is a section of the ruled surface $S = \mathbb{F}_n$. If $\mathcal{C}_{S^+/F^+} \cong \mathcal{O}_{\mathbb{F}_m}(s + (a + c)f)$, then $(C^2)_S = n + 2c$.

**Proof.** (i) The flopped variety $V^+$ is obtained by the blow-up $\sigma : W \to V$ along $S$ followed by the blow-down $\tau : W \to V^+$ of the exceptional divisor $E$ of $\sigma$ to the other direction, so that $E = \mathbb{P}[\mathcal{C}_{S/V}]$ is isomorphic to the fibre product $S \times_{\mathbb{P}^1} S^+ = \mathbb{P}[\pi^*\mathcal{O}(0, -m)]$, where $\pi : S = \mathbb{F}_n \to \mathbb{P}^1$ is the projection. Hence $\mathcal{C}_{S/V}$ is isomorphic to $\pi^*\mathcal{O}(0, -m) = \mathcal{O}_{\mathbb{F}_n}(0, -mf)$ modulo $\text{Pic}(S)$. Since $\mathcal{C}_{S/V}|_f \cong \mathcal{O}(1,1)$, we set $\mathcal{C}_{S/V} \cong \mathcal{O}_{\mathbb{F}_n}(s + af, s + (a - m)f)$ for an integer $a \in \mathbb{Z}$. Similarly, we put $\mathcal{C}_{S^+/V^+} \cong \mathcal{O}_{\mathbb{F}_m}(s + bf, s + (b - n)f)$. Now we shall show $a = b$. Let $\bar{s} \subset E$ be a curve which is mapped isomorphically onto the negative sections $s \subset S \cong \mathbb{F}_n$ and $s^+ \subset S^+ \cong \mathbb{F}_m$, i.e. $\bar{s} = s \times_{\mathbb{P}^1} s^+$. 

---

8
Then

\[(0.2.1) \quad (K_{V \cdot s}) = c_1(C_{S/V \cdot s}) + (K_{S \cdot s})
= (-n + a) + (-n + a - m) + (n - 2)
= 2a - n - m - 2,\]

\[(0.2.2) \quad (K_{V \cdot s^+}) = c_1(C_{S+/V \cdot s^+}) + (K_{S+ \cdot s^+})
= (-m + b) + (-m + b - n) + (m - 2)
= 2b - n - m - 2.\]

On the other hand, \((K_{V \cdot s}) = (K_{V \cdot s^+})\) because the canonical divisors are equal to \(K_W = \sigma^* K_V + E = \tau^* K_{V^+} + E\) and \((E \cdot \sigma^* s) = (E \cdot \tau^* s^+) = 0\). Hence \((0.2.1) = (0.2.2)\) implies \(a = b\).

(ii) The intersection in \(W\) of \(E\) and the proper transform of \(D\), is equal to \(P[C_{S/D}]\) in \(E = P[C_{S/V}]\). Then \(C^+ \subset S^+\) is isomorphic to \(P[C_{S/D} | s] \subset P[C_{S/V} | s]\). Since \(C_{S/V | s} \cong \mathcal{O}(-n + a, -n + a - m) \rightarrow C_{S/D} | s \cong \mathcal{O}(-n + a), \) i.e. \(\mathcal{O}(0, -m) \rightarrow \mathcal{O}(b),\) we see \((C^+)^2)_{S^+} = m + 2b\).

(iii) The flopped surface \(S^+\) is the exceptional divisor of the blow-up \(F^+ \rightarrow F\) along \(C\), so that \((S^+, f)_{F^+} = -1\) for a fibre \(f\) of \(S^+ = \mathbb{F}_m\). Hence we put \(C_{S+/F^+} \cong \mathcal{O}_{\mathbb{F}_m} (s + (a + c)_f)\) for an integer \(c \in \mathbb{Z}\). The intersection in \(W\) of \(E\) and the proper transform of \(F^+\) is equal to \(P[C_{S+/F^+}] \subset P[C_{S+/V^+}]\). Then \(C \subset S\) is isomorphic to \(P[C_{S+/F^+} | s] \subset P[C_{S+/V^+} | s]\). Since \(C_{S+/V^+} | s \cong \mathcal{O}(-m + a, -m + a - n) \rightarrow C_{S+/F^+} | s \cong \mathcal{O}(-m + a + c),\) i.e. \(\mathcal{O}(0, -n) \rightarrow \mathcal{O}(c),\) we see \((C^2)|_S = n + 2c\).

(0.3) Let \(T \cong \mathbb{P}^2\) be a subvariety of a smooth 4-fold \(V\) with \(C_{T/V} \cong \mathcal{O}_{\mathbb{P}^2}(1, 1)\). Then there are birational maps

\[V \xrightarrow{\sigma_1} V_1 \xrightarrow{\sigma_2} V_2,\]

where \(\sigma_1\) is the blow-up along \(T\) with the exceptional divisor \(E \cong \mathbb{P}^2 \times \mathbb{P}^1\), and \(\sigma_2\) is the blow-down of \(E\) onto \(\mathbb{P}^1\) (the projection to the second factor). Assume there is a smooth subvariety \(S\) of \(V\) of dimension 2 intersecting \(T\) transversely at one point \(p\). Then

(i) the birational transform \(S_2\) in \(V_2\) of \(S\) is the blow up of \(S\) at the point \(p = S \cap T\) with the exceptional line \(e = \sigma_2(E),\)

(ii) \(C_{S_2/V_2} \cong \sigma^*_2 \sigma^*_1 C_{S/V} \otimes \mathcal{O}_{S_2}(-e).\)
Conversely, let $S'$ be a smooth subvariety of a smooth four-fold $W$ with $C_{S'/W}|_e \cong \mathcal{O}(1,1)$ for a $(-1)$-curve $e$ on $S'$. Then, from the exact sequence $0 \to C_{S'/W}|_e \to C_{e/W} \to C_{e/S'} \to 0$, we see $C_{e/W} \cong \mathcal{O}(1,1,1)$. Hence there are birational maps

$$W \xrightarrow{\sigma_1} W_1 \xrightarrow{\sigma_2} W_2,$$

where $\sigma_1$ is the blow-up along $e$ with the exceptional divisor $E \cong \mathbb{P}^1 \times \mathbb{P}^2$, and $\sigma_2$ is the blow-down of $E$ onto $\mathbb{P}^2$ (the projection to the second factor). The birational transform $S'_2$ in $W_2$ of $S'$ is the blow-down of $S'$ along $e$ with $\sigma_1 \ast \sigma_2^* C_{S'_2/W_2} \cong C_{S'/W} \otimes \mathcal{O}_{S'}(e)$.

1. The birational map (I)

(1.1) In this section we consider the birational map (I) in Theorem. Let $\tau : V \to \text{Spec}(R)$ be the standard $\mathbb{P}^2$-bundle over the local ring $R$ of $X$ at a singular point of $\Delta$ constructed from the $R$-order $(f, g)_3, R$ of (0.1.3) (cf.§4). For the blow-up $\sigma : Y = Y_0 \cup Y_1 \to \text{Spec}(R)$ at the origin with $Y_0 = \text{Spec}(R[g/f])$ and $Y_1 = \text{Spec}(R[f/g])$, there is a standard $\mathbb{P}^2$-bundle $\tau_1 : W \to Y$ constructed from a maximal $\mathcal{O}_Y$-order $\Lambda$ with $\Lambda_{Y_0} = (f, g/f)_3, Y_0$ and $\Lambda_{Y_1} = (f/g, g)_3, Y_1$. In this section we decompose a birational map from $V$ to $W$ over $\text{Spec}(R)$ into elementary birational morphisms.

(1.2) Let $\sigma_1 : V_1 \to V$ be the blow-up at the vertex of the central fibre $\tau^{-1}(o)$ (the cone over the rational twisted cubic in $\mathbb{P}^3$, i.e. the surface contracted along the $(-3)$-curve on $\mathbb{F}_3$). Let $\sigma_2 : V_2 \to V_1$ be the blow-up along the proper transform $Q_1 \cong \mathbb{F}_3$ of $\tau^{-1}(o)$. We will prove the following Lemma in (4.2).

**Lemma.** $C_{\sigma_1(l)/V} \cong \mathcal{O}(2,0,-1)$ for any fibre $l$ of $Q_1 \cong \mathbb{F}_3$.

Assume the above Lemma. Since $\sigma_1$ is the blow-up at a point on $\sigma_1(l)$, Lemma implies $C_{l/V_1} \cong C_{\sigma_1(l)/V} \otimes \mathcal{O}(1) \cong \mathcal{O}(3,1,0)$. From the exact sequence $0 \to C_{Q_1/V_1}|_l \to C_{l/V_1} \to C_{l/Q_1} \cong \mathcal{O} \to 0$, we see

$$(1.2.1) \quad C_{Q_1/V_1}|_l \cong \mathcal{O}(3,1).$$

(1.3) Let $H_2 \subset V_2$ be the exceptional divisor of $\sigma_2$. From (1.2.1), the restriction $H_1 = \sigma_2^{-1}(l)$ of $\sigma_2 : H_2 \to Q_1$ to a fibre $l$ of $Q_1 \cong \mathbb{F}_3$ is
isomorphic to $\mathbb{F}_2$. Let $b_l$ be the $(-2)$-curve on $H_l$ and let $Q_2 \cong \mathbb{F}_3$ be the section of $\sigma_2 : H_2 \to Q_1$ defined by

\begin{equation}
Q_2 = \text{the union of } b_l's \text{ for all fibres } l \text{ of } Q_1 \cong \mathbb{F}_3.
\end{equation}

We define $\sigma_3$ as the blow up along $Q_2 \cong \mathbb{F}_3$.

**Lemma.** The exact sequence

\begin{equation}
0 \to C_{H_2/V_2}\mid Q_2 \to C_{Q_2/V_2} \to C_{Q_2/H_2} \to 0
\end{equation}

splits with isomorphisms $C_{H_2/V_2}\mid Q_2 \cong \mathcal{O}_{\mathbb{F}_3}(s-f)$ and $C_{Q_2/H_2} \cong \mathcal{O}_{\mathbb{F}_3}(2s+4f)$.

**Proof.** We will show the two isomorphisms in (1.3.2), i.e. $(H_2.s)_{V_2} = 4$, $(H_2.f)_{V_2} = -1$, $(Q_2.s)_{H_2} = 2$, $(Q_2.f)_{H_2} = -2$. Then Lemma follows from $\text{Ext}^1(\mathcal{O}_{\mathbb{F}_3}(2s+4f),\mathcal{O}_{\mathbb{F}_3}(s-f)) = H^1(\mathcal{O}_{\mathbb{F}_3}(-s-5f)) \cong H^1(\mathcal{O}_{\mathbb{F}_3}(-s)) = 0$ by Serre duality. We see $(H_2.f)_{V_2} = \mathcal{O}_{H_2}(-1)|_s = -1$ because the $(-2)$-curve $f = b_l$ on $H_l = \mathbb{P}[\mathcal{O}(3,1)]$ is defined by the surjection $\mathcal{O}(3,1) \to \mathcal{O}(1)$, and $(Q_2.f)_{H_2} = (Q_2|_{H_1}.b_l)_{H_1} = (b_l^2)_{E_2} = -2$ because $Q_2 \cap H_1 = b_l$. Let $E_1 = \mathbb{P}^3$ be the exceptional divisor of $\sigma_1$ and let $E_2$ be the proper transform of $E_1$ by $\sigma_2$. Then the restriction of $\sigma_2$ to $E_2$ is the blow-up of $E_1 = \mathbb{P}^3$ along the twisted cubic $C_1 = Q_1 \cap E_1$ with the exceptional divisor $S_2 = H_2 \cap E_2$ isomorphic to $\mathbb{F}_0$ because $C_{C_1/E_1} \cong \mathcal{O}(-5,-5)$. We will show in Lemma(1.5)(ii),

\begin{equation}
(1.3.3) \text{ the } (-3)-\text{curve } C_2 = Q_2 \cap S_2 \text{ on } Q_2 \cong \mathbb{F}_3 \text{ is a } (+2)-\text{curve on } S_2 \cong \mathbb{F}_0.
\end{equation}

If we assume (1.3.3), then $(Q_2.s)_{H_2} = (Q_2|_{S_2}.s)_{S_2} = (C_2^2)_{S_2} = 2$ and $(H_2.s)_{V_2} = \mathcal{O}_{H_2}(-1)|_s = \mathcal{O}_{S_2}(-1)|_{C_2} = 4$ because the $(+2)$-curve $C_2$ on $S_2 \cong \mathbb{F}_0$ is defined by a surjection $\mathcal{O}(-5,-5) \to \mathcal{O}(-4)$. \hfill $\square$

(1.4) For the twisted cubic $C_1$ in $E_1 = \mathbb{P}^3$, let $\phi : C_1 \to \mathbb{P}^1$ be the cyclic cover of degree three ramified at two points $\{p_0, p_\infty\} \subset \mathbb{P}^1$, and let $f_{p,0} \cup f_{p,1} \cup f_{p,2} \subset E_1$ be the three lines joining the three points $\phi^{-1}(p)$ for each point $p \in \mathbb{P}^1 - \{p_0, p_\infty\}$. Let

\[ M_1 = \text{the closure in } E_1 \text{ of } \bigcup_{p \in \mathbb{P}^1 - \{p_0, p_\infty\}} \bigcup_{i=0}^2 f_{p,i}. \]

We see the tangent lines at $p_0$ and $p_\infty$ to $C_1$ are contained in $M_1$. 

---

11
Lemma. (i) $M_1 \subset E_1 \cong \mathbb{P}^3$ is a non-normal quartic surface with multiplicity two along the twisted cubic $C_1$.

(ii) The proper transform $M_2$ of $M_1$ in $V_2$ is isomorphic to $\mathbb{F}_2$.

Proof. (i) Let $F(z_0, \ldots, z_3) = 0$ be the defining equation of $M_1$ in $\mathbb{P}^3$. For a point $p = (g:1) \in \mathbb{P}^1$, we assume $\phi^{-1}(p) = \{p_0, p_1, p_2\}$ with $p_i = (g: \omega^{2i} \beta^2 : \omega^i \beta : 1)$, $(i = 0, 1, 2)$ for $\beta^3 = g$. Then the three lines $f_{p,i} (i = 0, 1, 2)$ are contained in the plane $\{z_0 = g z_3\} \cong \mathbb{P}^2$, where the line $f_{p,i}$ is defined by the equation $l_i = z_1 + \omega^i \beta z_2 + \omega^{2i} \beta^2 z_3 = 0$. This means $F(g z_3, z_1, z_2, z_3)$ is divided by the product

$$l_0 l_1 l_2 = z_1^3 + g z_2^3 + g^2 z_3^3 - 3 g z_1 z_2 z_3$$

$$= z_1^3 + (z_0/z_3) z_2^3 + (z_0/z_3)^2 z_3^3 - 3 z_0 z_1 z_2.$$

Hence $F(z_0, \ldots, z_3)$ is equal to

$$(1.4.1) \quad F(z_0, \ldots, z_3) = z_1^3 z_3 + z_0 z_2^3 + z_0^2 z_3^2 - 3 z_0 z_1 z_2 z_3.$$  

We see easily the singular locus of $\{F = 0\}$ is equal to the twisted cubic $C_1 = \{z_0 z_2 - z_1^2 = z_0 z_3 - z_1 z_2 = z_1 z_3 - z_2^2 = 0\}$ with multiplicity two.

(ii) The quartic surface $M_1 = \{F = 0\}$ contains the line $s = \{z_0 = z_3 = 0\}$, so the conormal sheaf $\mathcal{C}_s/M_1$ is isomorphic to $\mathcal{O}(2)$ by the exact sequence $0 \rightarrow \mathcal{C}_s/M_1 \rightarrow \mathcal{C}_s/\mathbb{P}^3 \rightarrow \mathcal{C}_s/M_1 \rightarrow 0$. Since $s$ is disjoint from the singular locus $C_1$ of $M_1$ and since $M_2$ is nonsingular, we conclude $M_2$ is isomorphic to $\mathbb{F}_2$. □

(1.5) Next we investigate the 1-dimensional subscheme $M_2 \cap H_2$ in $V_2$.

Lemma. (i) $M_2 \cap H_2$ consists of two sections $C_2$ and $C_2'$ of $\sigma_2 : S_2 \cong \mathbb{F}_0 \rightarrow C_1$.

(ii) $C_2$ is linearly equivalent to $C_2'$ on both $S_2$ and on $M_2$, and $(C_2^2)_{S_2} = (C_2^2)_{M_2} = 2$.

Proof. $\sigma_2 : E_2 \rightarrow E_1$ is the blow-up of $E_1 \cong \mathbb{P}^3$ along the twisted cubic $C = \{z_0 z_2 - z_1^2 = z_0 z_3 - z_1 z_2 = z_1 z_3 - z_2^2 = 0\}$, so $E_2$ is defined by the equations

$$(1.5.1) \quad \begin{pmatrix} z_0 & z_1 & z_2 \\ z_1 & z_2 & z_3 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
in $E_1 \times \mathbb{P}^2$, where $(z_i)$ (resp. $(y_i)$) is the homogeneous coordinates of $E_1 = \mathbb{P}^3$ (resp. $\mathbb{P}^2$). The projection of $E_1 \times \mathbb{P}^2$ to $\mathbb{P}^2$ defines the $\mathbb{P}^1$-bundle structure $\pi : E_2 \to \mathbb{P}^2$ and $E_2 \cong \mathbb{P}[E]$ with the rank two vector bundle $E$ on $\mathbb{P}^2$ defined by

\begin{equation}
0 \to \mathcal{O}_{\mathbb{P}^2}(-1)^2 \xrightarrow{A} \mathcal{O}_{\mathbb{P}^2}^4 \to E \to 0,
\end{equation}

where

\[
A = \begin{pmatrix}
0 & y_0 & y_1 & 0 \\
y_0 & 0 & y_1 & y_2
\end{pmatrix},
\]

We see $E(-1)$ is a rank two stable vector bundle on $\mathbb{P}^2$ with $c_1(E(-1)) = 0$ and $c_2(E(-1)) = 2$. Since the equation (1.4.1) is equal to $F(z_0, ..., z_3) = -(z_0z_2 - z_1^2)(z_1z_3 - z_2^2) + (z_0z_3 - z_1z_2)^2$,

the proper transform $M_2 \subset E_2$ of $M_1$ is equal to the $\mathbb{P}^1$-bundle $\pi^{-1}(q)$ over the conic $q = \{y_0y_2 = y_1^2\}$ in $\mathbb{P}^2$, and $E|_q \cong \mathcal{O}_{\mathbb{P}^1}(3,1)$ because $M_2 = \mathbb{P}[E|_q] \cong \mathbb{P}_2$ and $c_1(E) = 2$ by (1.5.2). The intersection of $M_2$ and the exceptional divisor $S_2 = H_2 \cap E_2$ of $\sigma_2 : E_2 \to E_1$ is defined in $\mathbb{P}^3 \times \mathbb{P}^2$ by (1.5.1) together with

\[
rank \begin{pmatrix}
y_0 & y_1 \\
y_1 & y_2
\end{pmatrix} = 1, \quad rank \begin{pmatrix}
z_0 & z_1 & z_2 \\
z_1 & z_2 & z_3
\end{pmatrix} = 1,
\]

hence we see $M_2 \cap S_2 = M_2 \cap H_2$ consists of two sections $C_2$ and $C'_2$ over the twisted cubic $C_1 \subset E_1$, where

\begin{equation}
C_2 = \{(\lambda^3 : \lambda^2 \mu : \lambda \mu^2 : \mu^3) \times (\omega^2 \lambda^2 : \omega \lambda \mu : \mu^2)| (\lambda : \mu) \in \mathbb{P}^1\}
\end{equation}

and $C'_2$ is equal to $C_2$ replacing $\omega$ by $\omega^2$. Hence $C_2$ and $C'_2$ are linearly equivalent on both $S_2 \cong \mathbb{P}_0$ and $M_2 \cong \mathbb{P}_2$, and intersects at two points $(\lambda : \mu) = (1 : 0)$ and $(0 : 1)$. Therefore $(C'_2)_S_2 = (C_2)_M_2 = 2$ and $C_2$ is equal to $\mathbb{P}[O(3)]$ in $M_2 = \mathbb{P}[E|_q]$ by a surjection $E|_q \cong O(3,1) \to O(3)$. □

The intersection of $E_2$ and $Q_2$ (see (1.3.1)) is equal to one of $C_2$ and $C'_2$, say $C_2$. Let us consider the elementary transformation of $E$ along $C'_2 = E_2 \cap Q_2$:

\begin{equation}
0 \to E' \to E \to \mathcal{O}_q(3) \to 0,
\end{equation}

with $C_2 = \mathbb{P}[\mathcal{O}_q(3)]$ in $M_2 = \mathbb{P}[E|_q] \cong \mathbb{P}[O(3,1)]$ and $E' = \mathbb{P}[E']$.

(1.6) In the exact sequence (1.5.4) we will show
Lemma. $\mathcal{E}'|_q \cong \mathcal{O}(1, -1)$.

Proof. Let $(p_1, p_2) = (y_1/y_0, y_2/y_0)$ (resp. $(q_0, q_1) = (y_0/y_2, y_1/y_2)$) be the affine coordinates of the openset $U_0 = \{y_0 = 0\}$ (resp. $U_2 = \{y_2 = 0\}$) in $\mathbb{P}^2$. From the exact sequence (1.5.2), we see

\[
\begin{align*}
z_2 &= -(y_0/y_2)z_0 - (y_1/y_2)z_1, \\
z_3 &= -(y_0/y_2)z_1 - (y_1/y_2)z_2 \\
&= -(y_0/y_2)z_1 + (y_1/y_2)(y_0/y_2)z_0 + (y_1/y_2)z_1,
\end{align*}
\]

on $E_2 = \mathbb{P}[\mathcal{E}]$, so $\mathcal{E}$ has a free basis $\{z_2, z_3\}$ (resp. $\{z_1, z_0\}$) over $U_0$ (resp. $U_1$) with

\[
\begin{pmatrix}
z_2 \\
z_3
\end{pmatrix} = \begin{pmatrix}
-q_1 & -q_0 \\
q_1^2 - q_0 & q_0 q_1
\end{pmatrix} \begin{pmatrix}
z_1 \\
z_0
\end{pmatrix}.
\]

From (1.5.2), the kernel $\mathcal{E}'$ in (1.5.4) is given by

\[
\begin{align*}
\begin{pmatrix}
-w_2 \\
w_3
\end{pmatrix} &= \begin{pmatrix}1 & -\omega p_1 \\
p_2 - p_1^2 & 0
\end{pmatrix} \begin{pmatrix}z_2 \\
z_3
\end{pmatrix} \quad \text{on } U_0, \\
\begin{pmatrix}
w_1 \\
w_0
\end{pmatrix} &= \begin{pmatrix}1 & -\omega^2 q_1 \\
q_0 - q_1^2 & 0
\end{pmatrix} \begin{pmatrix}z_1 \\
z_0
\end{pmatrix} \quad \text{on } U_2.
\end{align*}
\]

Therefore $^t(w_2, w_3) = A^t(w_1, w_0)$ with $A$ equal to

\[
\begin{pmatrix}
1 & -\omega p_1 \\
0 & p_2 - p_1^2
\end{pmatrix} \begin{pmatrix}
-q_1 & -q_0 \\
q_1^2 - q_0 & q_0 q_1
\end{pmatrix} \begin{pmatrix}
1 & -\omega^2 q_1 \\
0 & q_0 - q_1^2
\end{pmatrix}^{-1}
\]

\[
= \begin{pmatrix}
1 & -\omega (q_1/q_0) \\
0 & (1/q_0) - (q_1/q_0)^2
\end{pmatrix} \begin{pmatrix}-q_1 & -q_0 \\
q_1^2 - q_0 & q_0 q_1
\end{pmatrix} \begin{pmatrix}
1 & -\omega^2 q_1 \\
0 & q_0 - q_1^2
\end{pmatrix}^{-1}
\]

\[
= \begin{pmatrix}
-q_1 + \omega(q_1/q_0)(q_0 - q_1^2) \\
-(q_0 - q_1^2)^2/q_0^2
\end{pmatrix} \begin{pmatrix}
-a_{12} \\
-\omega^2 q_1(q_0 - q_1^2)/q_0^2 + (q_1/q_0)
\end{pmatrix}
\]

where

\[
(q_0 - q_1^2)a_{12} = \{-q_1 + \omega(q_1/q_0)(q_0 - q_1^2)\} \omega^2 q_1 + (-q_0 - \omega q_1^2)
\]

\[
= -(q_0 - q_1^2) - (q_1^2/q_0)(q_1^2 - q_0),
\]

hence $a_{12} = -1 + (q_1^2/q_0)$. Since $q_0 = q_1^2$ on the conic $q$, we see

\[
A|_q = \begin{pmatrix}
-q_1 & 0 \\
0 & q_1^{-1}
\end{pmatrix}.
\]
This means $\mathcal{E}'|_q \cong \mathcal{O}(1, -1)$. □

(1.7) Let $F_3 \subset V_3$ be the exceptional divisor of the blow-up $\sigma_3$ of $V_2$ along $Q_2$. From Lemma(1.3), the restriction $F_i = \sigma_3^{-1}(b_i)$ of $F_3$ to a fibre $b_i$ of $Q_2 \cong \mathbb{F}_3$ is isomorphic to $\mathbb{F}_1$. Let $n_l$ be the $(-1)$-curve on $F_i$ and let $Q_3$ be the section of $\sigma_3 : F_3 \to Q_2$ defined by

\begin{equation}
Q_3 = \text{the union of } n_l \text{'s for all fibres } l \text{ of } Q_1 \cong \mathbb{F}_3.
\end{equation}

Since $\sigma_3 : E_3 \to E_2$ is the blow-up along $C_2 := Q_2 \cap E_2$ with the exceptional divisor $T_3 := F_3 \cap E_3$, Lemma(1.6) means $T_3 \cong \mathbb{P}[\mathcal{E}'|_q] \cong \mathbb{F}_2$.

\textbf{Lemma.} (i) $C_{Q_3/V_3}|_{n_l} \cong \mathcal{O}(1, 1)$,

(ii) the $(-3)$-curve $C_3 = Q_3 \cap E_3$ on $Q_3 \cong \mathbb{F}_3$ is the $(-2)$-curve on $T_3 = F_3 \cap E_3 \cong \mathbb{F}_2$, i.e. $C_3 \subset T_3$ is defined from the surjection $\mathcal{E}'|_q \cong \mathcal{O}(1, -1) \to \mathcal{O}(-1)$.

(iii) $Q_3$ is disjoint from $M_3$ in $V_3$.

\textbf{Proof.} (i) In the exact sequence

\begin{equation}
0 \to C_{F_3/V_3}|_{n_l} \to C_{Q_3/V_3}|_{n_l} \to C_{Q_3/F_3}|_{n_l} \to 0,
\end{equation}

we see $(F_3.n_l)_V = -\mathcal{O}_{F_3}(1)|_{n_l} = -1$ because $n_l$ is the $(-1)$-curve on $F_i \cong \mathbb{P}[\mathcal{O}(2, 1)]$ (see Lemma(1.3)), and $(Q_3.n_l)_F = (Q_3|_{F_i}, n_l)_F = (n_l)_E = -1$ because $Q_3 \cap F_i = n_l$.

(ii) From $(C_2^2)_S = 2$, we see $C_2 \subset S_2 \cong \mathbb{P}[C_{C_1/E_1}]$ is defined by a surjection $C_{C_1/E_1} \cong \mathcal{O}(-5, -5) \to \mathcal{O}(-4)$, so that $(S_2.C_2)_{E_2} = -\mathcal{O}_{S_2}(1)|_{C_2} = 4$. Therefore we obtain

\begin{equation}
0 \to C_{S_2/E_2}|_{C_2} \cong \mathcal{O}(-4) \to C_{C_2/E_2} \to C_{C_2/S_2} \cong \mathcal{O}(-2) \to 0,
\end{equation}

from which we have equalities

\begin{equation}
(T_3^2)_{E_2} = -\sigma_3^*C_2 + c_1(N_{C_2/E_2})r_3 = -\sigma_3^*C_2 + 6r_3, \quad (T_3^3)_{E_2} = -6
\end{equation}

for a fibre $r_3$ of $T_3 = \mathbb{F}_2 \to C_2$. Let $S_3 = H_3 \cap E_3$ be the proper transform in $E_3$ of $S_2$ and let

\[ m_3 = M_3 \cap E_3 = M_3 \cap T_3, \quad t_3 = H_3 \cap T_3 = S_3 \cap T_3 \]
be two sections of $\sigma_3 : T_3 \cong F_2 \to C_2$. We see from (1.7.4)

\[
(m_3.t_3)_{T_3} = (M_3|_{T_3}.S_3|_{T_3})_{T_3} = (M_3.S_3.T_3)_{E_3} \\
= (\sigma_3^*M_2 - T_3)(\sigma_3^*S_2 - T_3)T_3 \\
= \sigma_3^*(M_2.S_2)T_3 - \sigma_3^*(M_2 + S_2)T_3^2 + T_3^3 \\
= ((M_2 + S_2)(C_2).E_2 - 6.
\]

From (1.7.3) we see $(M_2.C_2)_{E_2} = (\sigma_2^*M_1 - 2S_2)C_2 = (M_1.C_1)_{E_1} - 2(S_2.C_2) = 4.3 - 2.4 = 4$. Hence $(m_3.t_3)_{T_3} = 4 + 4 - 6 = 2$. On the other hand, (1.3.2) induces the split exact sequence

\[
0 \to C_{H_2/V_2}|_f \cong O(1) \to C_{Q_2/V_2}|_f \to C_{Q_2/E_2}|_f \cong O(2) \to 0,
\]

for fibres $f = b_1$ of $Q_2 \cong F_2$. This implies $F_3 \cap H_3$ is covered by $(+1)$-curves $\mathbb{P}[C_{Q_2/H_2}|_f]$ in $\mathbb{P}[C_{Q_2/V_2}|_f] \cong F_1$, so $F_3 \cap H_3$ is disjoint from $Q_3$ (see (1.7.1)), hence $(C_3.t_3)_{T_3} = 0$. Thus the three sections $C_3$, $m_3$, $t_3$ of $T_3 \to C_2$ satisfy $(m_3.t_3)_{T_3} = 2$ and $(C_3.t_3)_{T_3} = 0$. This means $C_3$ is (resp. $m_3$ and $t_3$ are) the $(-2)$-curve (resp. $(+2)$-curves) on $T_3 \cong \mathbb{P}[O(1, -1)] \cong F_2$. Since $Q_3 \cap M_3$ is contained in $T_3 = F_3 \cap E_3$, (iii) follows from the fact that $C_3$ is disjoint from $m_3$. □

(1.8) The following Lemma implies $C_{M_3/V_3}|_f \cong O(1, 1)$ for fibres $f$ of $M_3 \cong F_2$, so there is a flop $V_3 \leftarrow V^+ \to M_3$.

**Lemma.** The two exact sequences

1. $0 \to C_{E_2/V_2}|_{M_2} \to C_{M_2/V_2} \to C_{M_2/E_2} \to 0,$
2. $0 \to C_{E_3/V_3}|_{M_3} \to C_{M_3/V_3} \to C_{M_3/E_3} \to 0$

split with the isomorphisms

\[
C_{E_2/V_2}|_{M_2} \cong O_{F_2}(s + 3f), \quad C_{M_2/E_2} \cong O_{F_2}(-4f), \\
C_{E_3/V_3}|_{M_3} \cong O_{F_2}(s + 3f), \quad C_{M_3/E_3} \cong O_{F_2}(2 - 2f).
\]

**Proof.** $C_{E_2/V_2}|_{M_2} \cong C_{E_3/V_3}|_{M_3} \cong O_{F_2}(s + 3f)$ follow from $(E_3.s)_{V_3} = (E_2.s)_{V_2} = (E_1.s)_{V_1} = O_{E_1}(-1)|_s = -1$ (since the image $\sigma_1(s)$ of $s$ in $V_1$ is a line on $E_1 = \mathbb{P}^3$), and $(E_3.f)_{V_3} = (E_2.f)_{V_2} = (E_1.f)_{V_1} = O_{E_1}(-1)|_f = -1$. We saw $\sigma_2(s)$ is disjoint from $C_1$ in the proof of
Lemma (1.4)(ii), so \((M_3.s)_{E_3} = (M_2.s)_{E_2} = (M_1.s)_{E_1} = \text{deg}(M_1) = 4\). Then the isomorphisms \(\mathcal{C}_{M_2/E_2} \cong \mathcal{O}_{E_2}(-4f)\) and \(\mathcal{C}_{M_3/E_3} \cong \mathcal{O}_{E_3}(-2f)\) follow from \((M_2.f)_{E_2} = (\sigma_2^* M_1 - 2S_2)f = (M_1.f)_{E_1} - 2(S_2.f)_{E_2} = 4 - 4 = 0\) and \((M_3.f)_{E_3} = (\sigma_3^* M_2 - T_3)f = (M_2.f) - (T_3.f)_{E_3} = 0 - 1 = -1\). Both (1.8.1) and (1.8.2) split because \(H^1(\mathcal{O}_{E_2}(s + 7f)) = H^1(\mathcal{O}_{E_2}(5f)) = 0\). □

(1.9) From Lemma (1.7)(i), (iii) and Lemma (1.8), we define \(V_4\) as the flopped variety of \(V_3\) along the disjoint union \(M_3\) and \(Q_3\). Since \(\mathcal{C}_{M_3/V_3} \cong \mathcal{O}_{F_3}(s + 3f, s - 2f)\) by (1.8.2), the flopped surface \(M_4\) in \(V_4\) of \(M_3\) is isomorphic to \(\mathbb{P}[\mathcal{C}_{M_3/V_3}/s] \cong \mathbb{F}_5\) with \(\mathcal{C}_{M_4/V_4} \cong \mathcal{O}_{F_5}(s + 3f, s + f)\) by Lemma (0.2)(i). Applying Lemma (0.2)(iii) to \(S = M_3\) and \(F = F_3\) we see \(\mathcal{C}_{M_4/F_4} \cong \mathcal{O}_{F_5}(s + 3f)\) because \(m_3 = F_3 \cap M_3\) is a (+2)-curve on \(M_3\) by Lemma (1.5)(ii). Hence there is a split exact sequence

(1.9.1)

\[ 0 \rightarrow \mathcal{C}_{F_4/V_4} \cong \mathcal{O}_{F_5}(s + f) \rightarrow \mathcal{C}_{M_4/V_4} \rightarrow \mathcal{C}_{M_4/F_4} \cong \mathcal{O}_{F_5}(s + 3f) \rightarrow 0. \]

(1.10) Let \(E_4, H_4, F_4\) be the birational transforms of \(E_3, H_3, F_3\), respectively. These are obtained as follows.

(a) \(E_4\) is constructed by the elementary transformation (1.5.4) and a blow up \(\epsilon_2\):

(1.10.1)

\[ E_2 = \mathbb{P}[\mathcal{E}] \overset{\sigma_3}{\rightarrow} E_3 \overset{\epsilon_1}{\rightarrow} \mathbb{P}[\mathcal{E}'] \overset{\epsilon_2}{\rightarrow} E_4, \]

where

(i) \(\sigma_3\) is the blow-up along the (+2)-curve \(C_2 = \mathbb{P}[\mathcal{O}_q(1)]\) in \(M_2 = \mathbb{P}[\mathcal{E}_q]\) with the exceptional divisor \(T_3 \cong \epsilon_1(T_3) = \mathbb{P}[\mathcal{E}'_q] \cong \mathbb{P}[\mathcal{O}(1, -1)]\),

(ii) \(\epsilon_1\) is the blow-down of the proper transform \(M_3\) of \(M_2 = \mathbb{P}[\mathcal{E}_q]\),

(iii) \(\epsilon_2\) is the blow-up along the (-2)-curve \(\epsilon_1(C_3) = \mathbb{P}[\mathcal{O}_q(-1)]\) in \(\mathbb{P}[\mathcal{E'}_q] \cong \mathbb{P}[\mathcal{O}(1, -1)]\) with the birational transform \(T_4\) in \(E_4\) of \(T_2\) isomorphic to \(\epsilon_2(T_4) = \mathbb{P}[\mathcal{E'}_q]\).

(b) \(H_4\) is the blow-up of \(H_3 \cong H_2\) along \(C_3' \cong \sigma_3(C_3') = C_2'\) with the exceptional divisor equal to the flopped surface \(M_4 = \mathbb{P}[\mathcal{C}_{M_3/V_3}/C_3'] \cong \mathbb{F}_5\).

(c) \(F_4\) is obtained from \(F_3 \cong F_2\) by

(1.10.2)

\[ F_3 \overset{\epsilon_3}{\rightarrow} F' \overset{\epsilon_4}{\rightarrow} F_4, \]

[--- 17 ---]
where

(iv) $\varepsilon_3$ is the blow-up along $m_3 = M_3 \cap F_3$ with the exceptional divisor equal to the flopped surface $M_4 = \mathbb{P}[C_{m_3/F_3}] \cong \mathbb{P}[C_{M_3/V_3}|m_3]$,

(v) $\varepsilon_4$ is the blow-down along $\varepsilon_3^{-1}(Q_3) \cong Q_3 \cong \mathbb{F}_3$.

The $\mathbb{P}^1$-bundle structure $\sigma_3 : F_3 \to Q_2$ induces a $\mathbb{P}^1$-bundle structure $\pi : F_4 \to M_4 \cong \mathbb{F}_5$.

(1.11) From (1.7) we see

**Proposition.** (i) The fibre $f_4$ of $\pi : F_4 \to M_4$ is an extremal rational curve on $V_4$ with $(-K_{V_4}.f_4) = 1$,

(ii) $E_4$ is mapped to $\mathbb{P}^1 \times \mathbb{P}^2$ by the contraction morphism $\sigma_4 : V_4 \to V_5$ of $f_4$,

(iii) The flopped surface $Q_4$ on $V_4$ of $Q_3$ is isomorphically mapped to $\mathbb{P}^1 \times q \subset \sigma_4(E_4) \cong \mathbb{P}^1 \times \mathbb{P}^2$ with a conic $q$ in $\mathbb{P}^2$.

**Proof.** In the exact sequence

$0 \to C_{F_4/V_4}|f_4 \to C_{f_4/V_4} \to C_{f_4/F_4} \cong \mathcal{O}(0,0) \to 0$, we will show $C_{F_4/V_4}|f_4 \cong \mathcal{O}(1)$, i.e. $(F_4,f_4)_{V_4} = -1$. The fibre $f_3$ of $T_3 = F_3 \cap E_3$ is isomorphically transformed to the fibres of $T_4 = F_4 \cap E_4$, so we may assume $f_4$ is contained in $E_4$. Hence $(F_4,f_4)_{V_4} = (F_4|E_4,f_4)_{E_4} = (T_4,f_4)_{E_4}$. We denote by $T' = \mathbb{P}[E'|q]$, by $Q_4$ the exceptional divisor of $\varepsilon_2$ of (1.9.1), and by $f'$, $q_4$, $m_3$ the fibre of $T'$, $Q_4$, $M_3$, respectively. Then we see

\[
(T_3.f_3)_{E_3} = (\varepsilon_3^*T' - M_3)(\varepsilon_1*f' - m_4) = (T'.f')_{E'} - 1,
\]

\[
(T_4.f_4)_{E_4} = (\varepsilon_2^*T' - Q_4)(\varepsilon_2*f' - q_4) = (T'.f')_{E'} - 1.
\]

Hence $(F_4,f_4)_{V_4} = (T_4,f_4)_{E_4} = (T_3.f_3)_{E_3} = -1$ because $T_3$ is the exceptional divisor of $\sigma_3 : E_3 \to E_2$.

(ii) We see from (i) that the image $\sigma_4(E_4)$ is equal to the result of the elementary transformation of $E' = \mathbb{P}[E']$ along $\varepsilon_1(C_3) = \mathbb{P}[O_q(-1)]$:

(1.11.1) $0 \to \mathcal{E}_5 \to \mathcal{E}' \to O_q(-1) \to 0$,

where $\sigma_4(E_4) = \mathbb{P}[\mathcal{E}_5]$ and $\varepsilon_1(C_3) = \mathbb{P}[O_q(-1)] \subset T' = \mathbb{P}[E'|q] \cong \mathbb{P}[O(1,-1)]$. Hence we will show $\mathcal{E}_5$ is isomorphic to $O_{\mathbb{P}^2}(-1)^2$. From (1.5.2) and (1.5.4), we see $c_1(\mathcal{E}') = 0$, $c_2(\mathcal{E}') = 2$ and $H^0(\mathcal{E}'(-4)) = 0$, so $h^0(\mathcal{E}'(1)) \geq \chi(\mathcal{E}'(1)) = 4$. Hence (1.11.1) implies $h^0(\mathcal{E}_5(1)) \geq 2$ and there is an inclusion $\iota : O_{\mathbb{P}^2}(-1)^2 \to \mathcal{E}_5$. On the other hand, $c_1(\mathcal{E}_5(1)) = c_2(\mathcal{E}_5(1)) = 0$, so the inclusion $\iota$ is an isomorphism.

— 18 —
(iii) The flopped surface $Q_4 = \mathbb{P}[\mathcal{C}_{Q_3/V_3}|C_3]$ on $V_4$ is equal to the exceptional divisor of the blow up of $\mathbb{P}[\mathcal{E}']$ along $\mathbb{P}[\mathcal{O}(-1)] \cong C_3$. Hence $Q_4$ is isomorphically mapped to $\mathbb{P}[\mathcal{E}_{[Q]}] \cong \mathbb{P}^1 \times q$ by the exact sequence (1). □

(1.12) Let $r_5 = \text{(point)} \times \text{(line)}$ in $E_5 = \sigma_4(E_4) \cong \mathbb{P}^1 \times \mathbb{P}^2$. To define $\sigma_5 : V_5 \to V_6 = W$ we show

**Lemma.** (i) There is a split exact sequence

$$0 \to C_{E_5/V_5}|_{r_5} \cong \mathcal{O}(1) \to C_{r_5/V_5} \to C_{r_5/E_5} \cong \mathcal{O}(0,-1) \to 0,$$

(ii) $r_5$ is an extremal rational curve on $V_5$ and the associated morphism $\sigma_5 : V_5 \to V_6$ contracts $E_5 \cong \mathbb{P}^1 \times \mathbb{P}^2$ onto the first factor $\mathbb{P}^1$.

**Proof.** (i) We show $C_{E_5/V_5}|_{r_5} \cong \mathcal{O}(1)$, i.e. $(E_5.r_5)V_5 = -1$. The surface $S_2 = H_2 \cap E_2 \cong \mathbb{F}_0$ in $V_2$ is transformed isomorphically onto $S_4 = H_4 \cap E_4$ in $V_4$ and the Stein factorization of the composite $S_2 \cong S_4 \subset E_4 \xrightarrow{r_4} E_5 \cong \mathbb{P}^1 \times \mathbb{P}^2 \xrightarrow{\pi_2} \mathbb{P}^1$ is given by

$$S_2 \xrightarrow{\phi} C_1 \xrightarrow{\phi} \mathbb{P}^1,$$

where $\phi$ is the associated cyclic cover of degree three in (1.4). Let $r_4$ be the isomorphic image in $S_4$ of a fibre $r_2$ of $\sigma_1 : S_2 \cong \mathbb{F}_0 \to C_1$. Then $r_5 = \sigma_4(r_4)$ is equal to $\text{(point)} \times \text{(line)}$ in $E_5 = \mathbb{P}^1 \times \mathbb{P}^2$. Hence $(E_4.r_4)V_4 = (\sigma_4^*E_5.r_4)V_4 = (E_5.r_5)V_5$. The left-hand side is equal to $(E_4.r_4)V_4 = (E_4|_{H_4}.r_4)H_4 = (S_4.r_4)H_4$. We recall $M_2 \cap H_2 = M_2 \cap S_2 = C_2 \cup C'_2$ (Lemma(1.5)), and $H_4 \cdots \to H_2$ is the blow-up along $C'_2$ with the exceptional divisor $M_4$ (see (1.10)(b)). Hence $(S_4.r_4)H_4 = (\sigma^*S_2 - M_4)r_4 = (S_2.r_2)H_2 - (M_4.r_4)H_4 = 0 - 1 = -1$ because $S_2 \cong \mathbb{F}_0$ is a $\mathbb{P}^1$-bundle over the twisted cubic $C_1$ with a fibre $r_2$. (ii) follows from (i). □

(1.13) Let $h_6$ be the image in $V_6 = W$ of a fibre $h_2$ of the $\mathbb{P}^1$-bundle $\sigma_1 : H_2 \to Q_1 \cong \mathbb{F}_3$. To see $W$ is a standard $\mathbb{P}^2$-bundle over the blow-up $Y$ at the origin of $\text{Spec}(R)$, we show

**Lemma.** $(-K_W.h_6)_W = 1$.

**Proof.** We assume the birational transform $h_4$ on $V_4$ of $h_2$ is disjoint from $S_4 = H_4 \cap E_4$. Then $(H_4.h_4)V_4 = (H_3.h_3)V_3 = (\sigma^*H_2 - F_3)h_3 = \ldots$
\[(H_2.h_2)_V - 1 = -2\] because \(H_2\) is the exceptional divisor of \(\sigma_1\) with the fibre \(h_2\). Hence, from the exact sequence

\[0 \to C_{H_2/V_2|h_2} \simeq \mathcal{O}(2) \to C_{h_2/V_2} \to C_{h_2/H_2} \simeq \mathcal{O}(0,0) \to 0,\]

we see \(C_{h_2/V_2} \simeq \mathcal{O}(2,0,0)\) and \((-K_{V_2}h_2) = 0\). On the other hand, from \((F_4.h_4)_V = 0\) and \((E_5.h_5)_V = 0\), we see

\[(-K_{V_4}h_4)_V = (\sigma_4^*(-K_{V_5}) - F_4)h_4 = (-K_{V_5}h_5)_V - 1\]
\[= (\sigma_5^*(-K_{V_6}) - 2E_5)h_5 - 1 = (-K_{V_6}h_6) - 1.\]

Therefore \((-K_{V_6}h_6) = 1\). \(\square\)

(1.14) Next we determine the conormal bundles \(C_{Q_3/V_3}\) and \(C_{M_5/V_5}\).

**Lemma.** (i) There is an exact sequence

\[0 \to C_{F_3/V_3|Q_3} \simeq \mathcal{O}_{F_3}(s - f) \to C_{Q_3/V_3} \to C_{Q_3/F_3} \simeq \mathcal{O}_{F_3}(s + 5f) \to 0,\]

(ii) \(C_{Q_3/V_3} \simeq \mathcal{O}_{F_3}(s + 2f, s + 2f)\), \(C_{M_5/V_5} \simeq \mathcal{O}_{F_5}(s + f, 2s + 4f)\).

**Proof.** (i) Let \(f\) (resp. \(s\)) be a fibre (resp. the \((-3)\)-curve) on \(Q_3 \simeq \mathbb{F}_3\). For the isomorphism \(C_{F_3/V_3|Q_3} \simeq \mathcal{O}_{F_2}(s - f)\), we show \((F_3.f)_V = -1\) and \((F_3.s)_V = 4\). By the definition (1.7.1), \(f\) is the \((-1)\)-curve on \(F_3 = \text{Proj}[C_{Q_2/V_2}|l] = \mathcal{O}(1,2)\) by (1.3.2). Hence \((F_3.f)_V = \mathcal{O}_{F_2}(-1)|_f = -1\). We saw in Lemma(1.7)(ii) that \(s = C_3\) is the \((-1)\)-curve on \(T_3 = \text{Proj}[C_{Q_2/V_2}|c_3]\) with \(C_{Q_2/V_2}|c_2 \simeq \mathcal{O}(4, -2)\) by (1.3.2). Hence \((Q_3.s)_V = \mathcal{O}_{F_3}(-1)|_s = \mathcal{O}_{T_3}(-1)|_{C_3} = 4\). Similarly, \((Q_3.f)_V = (Q_3|F_1.f)_V = (f^2)_V = -1\) and \((Q_3.s)_F = (Q_3|T_3.C_3)_T = (C_3^2)_T = -2\), hence \(C_{Q_3/F_3} = \mathcal{O}_{F_3}(s + 5f)\).

(ii) Since \(Q_4 \simeq \mathbb{F}_0\) by Lemma(1.10)(iii), \(C_{Q_3/V_3} \simeq \mathcal{O}_{F_3}(s + af, s + af)\) for an integer \(a \in \mathbb{Z}\). The exact sequence proved in (i) implies \(a = 1\).

The fibre \(f_4\) of \(M_4 \simeq \mathbb{F}_5\) is the \((-1)\)-curve on \(\text{Proj}[C_{M_5/V_5}|/f] = \mathbb{F}_1\) with \(f = \sigma_4(f_4)\), so that \(f_4 = \text{Proj}[C_{M_5/V_5}|/f]\) is defined by the surjection \(\mathcal{O}(a, a + 1) \to \mathcal{O}(a)\) for an integer \(a \in \mathbb{Z}\). Here \(a = \mathcal{O}_{F_4}(1)|_{f_4} = C_{F_4/V_4|/f_4} = 1\) by (1.9.1). Next we apply Lemma(0.2)(ii) to \(S = M_3, D = E_3\) and \(C^+ = s_4 := M_4 \cap E_4\). Then \(C_{M_3/E_3} = \mathcal{O}_{E_2}(s - 2f)\) in (1.8.2) implies \((s_4^2)_M = 5 - 2.5 = -5\). This \((-5)\)-curve \(s_4\) on \(M_4 = \mathbb{F}_5\) is the \((+2)\)-curve on \(\text{Proj}[C_{M_5/V_5}|/s] = E_4 \cap F_4 \simeq \mathbb{F}_2\) with \(s = \sigma_4(s_4)\), so that \(s_4 \subset \text{Proj}[C_{M_5/V_5}|/s]\) is defined by a surjection \(\mathcal{O}(b - 2, b) \to \mathcal{O}(b)\)
for an integer $b \in \mathbb{Z}$. Here $b = \mathcal{O}_{F_4}(1)|_{s_4} = C_{F_4/V_4}|_{s} = -4$ by (1.9.1). Thus we see

$$(1.14.1) \quad C_{M_5/V_5}|_{f} = \mathcal{O}(1,2), \quad C_{M_5/V_5}|_{s} = \mathcal{O}(-6,-4).$$

Now the canonical surjection $\sigma^*_4 C_{M_5/V_5} \to \mathcal{O}_{F_4}(1)$ induces the surjection $\phi : \sigma^*_4 C_{M_5/V_5}|_{M_4} = C_{M_5/V_5} \to \mathcal{O}_{F_4}(1)|_{M_4} \cong C_{F_4/V_4}|_{M_4} = \mathcal{O}_{F_5}(s + f)$ by (1.9.1). Then (1.14.1) implies $\text{Ker}(\phi) = \mathcal{O}_{F_5}(2s + 4f)$ and there is an exact sequence

$$(1.14.2) \quad 0 \to \mathcal{O}_{F_5}(2s + 4f) \to C_{M_5/V_5} \xrightarrow{\phi} C_{F_4/V_4}|_{M_4} = \mathcal{O}_{F_5}(s + f) \to 0.$$

Here $H^1(\mathcal{O}_{F_5}(s + 3f)) = H^1(\mathcal{O}(3,-2)) \cong k$, but $C_{M_5/V_5}|_{s} = \mathcal{O}(-6,-4)$ means that the restriction of (1.14.2) to $s$ splits. Hence (1.14.2) itself splits and $C_{M_5/V_5} \cong \mathcal{O}_{F_5}(s + f, 2s + 4f)$. □

(1.15) Let $C_4 = Q_4 \cap F_4$, $C_5 = \sigma_4(C_4)$ and $C_6 = \sigma_5(C_5) = \sigma_5(E_5)$. We show

Lemma. $C_{C_6/V_6} \cong \mathcal{O}(1,1,1)$.

Proof. We apply Lemma(0.2)(iii) to $S = Q_3$ and $F = E_3$. Since $C_3 = Q_3 \cap E_3$ is the (-3)-curve on $Q_3 = F_3$, we see $C_{Q_4/E_4} \cong \mathcal{O}_{F_0}(s - f)$. On the other hand, $C_{Q_3/V_3} \mathcal{O}_{F_3}(s + 2f, s + 2f)$ in Lemma(1.14)(ii) means $C_{Q_4/V_4} \cong \mathcal{O}_{F_0}(s + 2f, s - f)$ by Lemma(0.2)(i). Hence, from the exact sequence

$$(1.15.1) \quad 0 \to C_{E_4/V_4}|_{Q_4} \to C_{Q_4/V_4} \to C_{Q_4/E_4} \to 0,$$

with the isomorphisms $C_{Q_4/V_4} \cong \mathcal{O}_{F_0}(s + 2f, s - f)$ and $C_{Q_4/E_4} \cong \mathcal{O}_{F_0}(s - f)$, we obtain $C_{E_4/V_4}|_{Q_4} \cong \mathcal{O}_{F_0}(s + 2f)$. Therefore $C_{E_5/V_5}|_{C_5} \cong \mathcal{O}_{F_0}(s + 2f)|_{s + 3f} = \mathcal{O}(5)$. We saw in (1.4.1) that $C_{C_5/E_5} \cong C_{M_5/V_5}|_{C_5} = \mathcal{O}(-6,-4)$. Hence the exact sequence $0 \to C_{E_5/V_5}|_{C_5} = \mathcal{O}(5) \to C_{C_5/V_5} \to C_{C_5/E_5} = \mathcal{O}(-6,-4) \to 0$ implies $(K_{V_5}.C_5) = 5 - 10 - 2 = -7$. Since $\sigma_5 : C_5 \to C_6$ is the cyclic cover of degree three (cf.(1.4)), $(K_{V_5}.C_5) = (\sigma^*_5 K_{V_5} + 2E_5)C_5 = 3(K_{V_5}.C_6) + 2(E_5.C_5)$, hence $(K_{V_6}.C_6) = +1$. We saw $E_5 = \mathbb{P}[C_{C_6/V_6}]$ is isomorphic to $F_0$ in Lemma(1.11)(i), so $C_{C_6/V_6} = \mathcal{O}(a, a, a)$ for an integer $a \in \mathbb{Z}$. then $(K_{V_6}.C_6) = 1$ means $a = 1$. □
(1.16) We write down the conormal bundles of the ruled surfaces $Q, M, T, S$.

\[
\begin{align*}
C_{Q_2 / V_2} &= \mathcal{O}_{F_3}(s - f, 2s + 4f), & C_{Q_3 / V_3} &= \mathcal{O}_{F_3}(2s + 4f, 2s + 4f), \\
C_{Q_4 / V_4} &= \mathcal{O}_{F_0}(s + 2f, s - f), & C_{Q_5 / V_5} &= \mathcal{O}_{F_0}(s + 2f, -2f), \\
C_{M_2 / V_2} &= \mathcal{O}_{F_2}(s + 3f, -4f), & C_{M_3 / V_3} &= \mathcal{O}_{F_2}(s + 3f, s - 2f), \\
C_{M_4 / V_4} &= \mathcal{O}_{F_5}(s + f, s + 3f), & C_{M_5 / V_5} &= \mathcal{O}_{F_5}(2s + 4f, s + f), \\
C_{T_3 / V_3} &= \mathcal{O}_{F_2}(s - 2f, 3f), & C_{T_4 / V_4} &= \mathcal{O}_{F_2}(s - 4f, 5f), \\
C_{S_2 / V_2} &= \mathcal{O}_{F_0}(3f, s - 5f), & C_{S_3 / V_3} &= \mathcal{O}_{F_0}(3f, 2s - 5f), \\
C_{S_4 / V_4} &= \mathcal{O}_{F_0}(s + 3f, s - 5f).
\end{align*}
\]

(1.17) The effective cones of $V_i$ over $X = \text{Spec}(R)$ ($1 \leq i \leq 6$) and the intersection numbers with generators of the Picard group, are given as follows. Here $q_i, m_i, s_i, t_i$ are the fibres of the ruled surfaces $Q_i, M_i, S_i, T_i$, respectively.

1. \(\text{NE}(V_1 / X) = \mathbb{R}[q_1] \oplus \mathbb{R}[m_1], \quad \text{Pic}(V_1 / X) = \mathbb{Z}(-K_{V_1}) \oplus \mathbb{Z}E_1.\)
   
   \[
   (-K_{V_1}.q_1) = -2, \quad (-K_{V_1}.m_1) = +3,
   \]
   
   \[
   (E_1.q_1) = +1, \quad (E_1.m_1) = -1.
   \]

2. \(\text{NE}(V_2 / X) = \mathbb{R}[q_2] \oplus \mathbb{R}[m_2] \oplus \mathbb{R}[s_2], \quad \text{Pic}(V_2 / X) = \mathbb{Z}(-K_{V_2}) \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}H_2.\)

   \[
   \begin{array}{|c|c|c|c|}
   \hline
   & q_2 & m_2 & s_2 \\
   \hline
   -K_{V_2} & -1 & 1 & 1 \\
   E_2 & 1 & 2 & 0 \\
   H_2 & -1 & 2 & -1 \\
   \hline
   \end{array}
   \]

3. \(\text{NE}(V_3 / X) = \mathbb{R}[q_3] \oplus \mathbb{R}[m_3] \oplus \mathbb{R}[s_3] \oplus \mathbb{R}[t_3], \quad \text{Pic}(V_3 / X) = \mathbb{Z}(-K_{V_3}) \oplus \mathbb{Z}E_3 \oplus \mathbb{Z}H_3 \oplus \mathbb{Z}F_3.\)

   \[
   \begin{array}{|c|c|c|c|c|}
   \hline
   & q_3 & m_3 & s_3 & t_3 \\
   \hline
   -K_{V_3} & 0 & 0 & 0 & 1 \\
   E_3 & 1 & 2 & 0 & 0 \\
   H_3 & 0 & 1 & -1 & 1 \\
   F_3 & -1 & 1 & 1 & -1 \\
   \hline
   \end{array}
   \]
(4) \[ \text{NE}(V_4/X) = \mathbb{R}[q_4] \oplus \mathbb{R}[m_4] \oplus \mathbb{R}[s_4] \oplus \mathbb{R}[t_4], \]
\[ \text{Pic}(V_4/X) = \mathbb{Z}(-K_{V_4}) \oplus \mathbb{Z}E_4 \oplus \mathbb{Z}H_4 \oplus \mathbb{Z}F_4. \]

\[
\begin{array}{|c|c|c|c|}
\hline
& q_4 & m_4 & s_4 & t_4 \\
\hline
-K_{V_4} & 0 & 0 & 0 & 1 \\
E_4 & -1 & 1 & -1 & 0 \\
H_4 & 0 & 5 & 2 & 1 \\
F_4 & 1 & -1 & 2 & -1 \\
\hline
\end{array}
\]

(5) \[ \text{NE}(V_5/X) = \mathbb{R}[q_5] \oplus \mathbb{R}[m_5] \oplus \mathbb{R}[s_5], \]
\[ \text{Pic}(V_5/X) = \mathbb{Z}(-K_{V_5}) \oplus \mathbb{Z}E_5 \oplus \mathbb{Z}H_5. \]

\[
\begin{array}{|c|c|c|}
\hline
& q_5 & m_5 & s_5 \\
\hline
-K_{V_5} & 1 & -1 & 2 \\
E_5 & -1 & 1 & -1 \\
H_5 & 1 & 3 & 2 \\
\hline
\end{array}
\]

(6) \[ \text{NE}(V_6/X) = \mathbb{R}[q_6] \oplus \mathbb{R}[m_6], \quad \text{Pic}(V_6/X) = \mathbb{Z}(-K_{V_6}) \oplus \mathbb{Z}H_6. \]

\[
\begin{array}{|c|c|}
\hline
& q_6 & m_6 \\
\hline
-K_{V_6} & -1 & 1 \\
H_6 & -2 & 0 \\
\hline
\end{array}
\]

Here \( q_6 = \sigma_5(q_5) \) is equal to \( C_6 \) in (1.15), and \( m_6 = \sigma_5(m_5) \).

2. The birational map (II)

(2.1) In this section we consider the birational map (II). Let \( \tau : V \to \text{Spec}(R) \) be a standard \( \mathbb{P}^2 \)-bundle over the local ring \( R \) of \( X \) at a smooth point of the non-smooth locus constructed from the \( R \)-order (0.1.2). Let \( \tau^{-1}(o) = R \cup S \cup T \) be the central fibre such that \( R, S, T \) isomorphic to \( \mathbb{F}_1 \), and

\[ r = R \cap T, \quad s = S \cap R, \quad t = T \cap S \]
are the \((-1)\)-curves on \(R, S, T\), respectively. For the the non-smooth locus \(\Delta \subset \text{Spec}(R)\) we assume \(\tau^{-1}(\Delta)\) decomposes into three divisors \(D, H, L\) of \(V\) such that
\[
R = \tau^{-1}(o) \cap D, \quad S = \tau^{-1}(o) \cap H, \quad T = \tau^{-1}(o) \cap L.
\]
Then \(V\) is obtained by twice blow-ups of \(\mathbb{P}^2_R\) along \(\mathbb{P}^1_\Delta \supset \mathbb{P}^0_\Delta \cong \Delta\):
\[
V \xrightarrow{\sigma} V_0 \xrightarrow{\sigma_0} \mathbb{P}^2_R,
\]
where \(\sigma_0\) is the blow-up along \(\mathbb{P}^1_\Delta\) with the exceptional divisor \(H_0\), and \(\sigma\) is the blow-up along \(L_0 = \sigma_0^{-1}(\mathbb{P}^0_\Delta)\). Then \(D\) (resp. \(H\)) is the proper transform of \(\mathbb{P}^2_\Delta\) (resp. \(H_0\)) and \(L\) is the exceptional divisor of \(\sigma\).

\[
\begin{array}{cccc}
V & D & H & L \\
\sigma & \downarrow & \downarrow & \downarrow \\
V_0 & D_0 & H_0 & \supset L_0 \\
\sigma_0 & \downarrow & \downarrow & \downarrow \\
\mathbb{P}^2_R \supset \mathbb{P}^2_\Delta \supset \mathbb{P}^1_\Delta \supset \mathbb{P}^0_\Delta \cong \Delta
\end{array}
\]

(2.2) The center \(L_0 = \sigma_0^{-1}(\mathbb{P}^0_\Delta)\) of the blow-up \(\sigma : V \to V_0\) is isomorphic to \(\mathbb{P}[[C_{\Phi_1}]_{|L_0}] \cong \mathbb{P}^1_\Delta\).

**Lemma.** (i) \(C_{L_0/V_0} \cong \mathcal{O}_{\mathbb{P}^1_\Delta}(1, 0)\), (ii) There is a split exact sequence
\[
0 \to C_{L/V} \mid T \cong \mathcal{O}_{\mathbb{P}^1}(s + f) \to C_{T/V} \to C_{T/L} \cong \mathcal{O}_{\Phi_1} \to 0.
\]

**Proof.** (i) We see \(C_{H_0/V_0} \mid L_0 \cong \mathcal{O}_{H_0}(1) \mid L_0 \cong \mathcal{O}_{L_0}(1) \cong \mathcal{O}_{\mathbb{P}^1_\Delta}(1)\) and \(C_{L_0/H_0} \cong \sigma_0^* C_{\mathbb{P}^2_\Delta/\mathbb{P}^0_\Delta} \cong \mathcal{O}_{L_0}\). Hence (i) follows from the exact sequence \(0 \to C_{H_0/V_0} \mid L_0 \to C_{L_0/V_0} \to C_{L_0/H_0} \to 0\).

(ii) Let \(p = \mathbb{P}^0_\Delta \cap \tau^{-1}(o)\) and let \(f = \sigma_0^{-1}(p) \cong \mathbb{P}^1_k\) be the fibre of the \(\mathbb{P}^1\)-bundle \(\sigma_0 : L_0 \to \mathbb{P}^0_\Delta \cong \Delta\). Then \(C_{f/L_0} \cong \mathcal{O}\) and \(C_{L_0/V_0} \mid f \cong \mathcal{O}(1, 0)\) by (i). Hence the exact sequence \(0 \to C_{L_0/V_0} \mid f \to C_{f/V_0} \to C_{f/L_0} \to 0\) splits. The surjection \(\phi : \sigma^*(C_{L_0/V_0} \mid f) \cong (\sigma^* C_{L_0/V_0}) \mid T \to C_{L/V} \mid T\) induces a commutative diagram with exact rows:
\[
\begin{array}{cccccc}
0 \to \sigma^*(C_{L_0/V_0} \mid f) & \rightarrow & \sigma^* C_{f/V_0} & \rightarrow & \sigma^* C_{f/L_0} & \rightarrow & 0 \\
\phi & \downarrow & \downarrow & & \downarrow & & \\
0 \to C_{L/V} \mid T & \rightarrow & C_{T/V} & \rightarrow & C_{T/L} & \rightarrow & 0.
\end{array}
\]
Since the first row splits, the second row also splits. Hence, for the proof of (ii), we will show $\mathcal{C}_{L/V}|_T \cong \mathcal{O}_{\mathbb{F}_1}(s + f)$. The exceptional divisor $L$ of $\sigma$ is equal to $\mathbb{P}[\mathcal{C}_{L_0/V_0}]$ with $\mathcal{C}_{L_0/V_0} \cong \mathcal{O}_{\mathbb{F}_1}(1, 0)$ by (i), so that $T = \mathbb{P}[\mathcal{C}_{L_0/V_0}|_f] \cong \mathbb{P}[\mathcal{O}(1, 0)]$ and $\mathcal{C}_{L/V}|_T \cong \mathcal{O}_L(1)|_T \cong \mathcal{O}_{\mathbb{F}_1}(s + f)$. □

By symmetry of $R, S, T$, (ii) implies $\mathcal{C}_{R/V} \cong \mathcal{C}_{S/V} \cong \mathcal{C}_{T/V} \cong \mathcal{O}_{\mathbb{F}_1}(s + f, 0)$.

(2.3) Let $\sigma_1 : V_1 \rightarrow V$ be the blow-up along $R$ with the exceptional divisor $E_1$ and the proper transforms $D_1, H_1, L_1, S_1, T_1$ of $D, H, L, S, T$, respectively. Let $M_1$ (resp. $N_1$) be the exceptional divisor of the restriction $\sigma_1 : H_1 \rightarrow H$ (resp. $\sigma_1 : L_1 \rightarrow L$) of $\sigma_1$ and let

$$s_1 = M_1 \cap S_1 = \mathbb{P}[\mathcal{C}_{s/H}], \quad r_1 = N_1 \cap T_1 = \mathbb{P}[\mathcal{C}_{r/L}].$$

Lemma. (i) $M_1 \cong \mathbb{F}_1, (s_1^2)_{M_1} = 1$, (ii) $N_1 \cong \mathbb{F}_0, (r_1^2)_{N_1} = 0$.

Proof. We see $(S.s)_H = (T.r)_L = 0, (s_1^2)_S = -1, (r_1^2)_{N_1} = 0$, hence we have exact sequences

$$0 \rightarrow \mathcal{C}_{S/H}|_s \cong \mathcal{O} \rightarrow \mathcal{C}_{s/H} \rightarrow \mathcal{C}_{s/S} \cong \mathcal{O}(1) \rightarrow 0,$$

$$0 \rightarrow \mathcal{C}_{T/L}|_r \cong \mathcal{O} \rightarrow \mathcal{C}_{r/L} \rightarrow \mathcal{C}_{r/T} \cong \mathcal{O} \rightarrow 0.$$

Lemma follows since $s_1 \subset M_1$ (resp. $r_1 \subset N_1$) is defined by the surjection $\mathcal{C}_{s/H} \rightarrow \mathcal{C}_{s/S}$ (resp. $\mathcal{C}_{r/L} \rightarrow \mathcal{C}_{r/T}$). □

(2.4) For the proper transforms $S_1$ (resp. $T_1$) of $S$ (resp. $T$), we have

Lemma. The exact sequences

(2.4.1) $0 \rightarrow \mathcal{C}_{H_1/V_1}|_s \rightarrow \mathcal{C}_{S_1/V_1} \rightarrow \mathcal{C}_{S_1/H_1} \rightarrow 0$,

(2.4.2) $0 \rightarrow \mathcal{C}_{L_1/V_1}|_t \rightarrow \mathcal{C}_{T_1/V_1} \rightarrow \mathcal{C}_{T_1/L_1} \rightarrow 0$.

splits with isomorphisms $\mathcal{C}_{H_1/V_1}|_s \cong \mathcal{C}_{L_1/V_1}|_t \cong \mathcal{O}_{\mathbb{F}_1}(s + f), \mathcal{C}_{T_1/H_1} \cong \mathcal{O}_{\mathbb{F}_1}(s)$ and $\mathcal{C}_{T_1/L_1} \cong \mathcal{O}_{\mathbb{F}_1}(f)$.

Proof. We saw $\mathcal{C}_{H_1/V_1}|_s \cong \mathcal{C}_{H/V}|_s \cong \mathcal{O}_{\mathbb{F}_1}(s + f)$ and $\mathcal{C}_{L_1/V_1}|_t \cong \mathcal{C}_{L/V}|_T \cong \mathcal{O}_{\mathbb{F}_1}(s + f)$ in Lemma(2.2)(ii). Hence we show $\mathcal{C}_{S_1/V_1} \cong \mathcal{O}_{\mathbb{F}_1}(s)$ and $\mathcal{C}_{T_1/L_1} \cong \mathcal{O}_{\mathbb{F}_1}(f)$, i.e. $(S_1.f)_{H_1} = -1, (T_1.f)_{L_1} = 0, (S_1,s_1)_{H_1} = 1$ and $(T_1,t_1)_{L_1} = -1$. For, $(S_1.f)_{H_1} = (\sigma_1^*S - M_1)f = (S.f)_H - (M_1.f)_{H_1} = 0 - 1 = -1$. By Lemma(2.3)(i), $s_1 \subset M_1$ is
defined by a surjection \( C_R/V | f \cong \mathcal{O}(1,0) \to \mathcal{O}(1) \), so \((M_1.s_1)_{H_1} = \mathcal{O}_{M_1}(-1)|_{s_1} = -1\), hence \((S_1.s_1)_{H_1} = (\sigma_1^*S - M_1)s_1 = (S.s)_{H} - (M_1.s_1)_{H_1} = 0 - (-1) = 1\). Similarly, \((T_1.f)_{L_1} = (\sigma_1^*T - N_1)f = (T.f)_L - (N_1.f)_{L_1} = 0 - 0 = 0\) and \((T_1.t_1)_{L_1} = (\sigma_1^*T - N_1)t_1 = (T.t)_L - (N_1.t_1)_{L_1} = 0 - 1 = -1\). The splitting of (2.4.1) and (2.4.2) follows from \(\text{Ext}(\mathcal{O}_{F_1}(s),\mathcal{O}_{F_1}(s + f)) = H^1(\mathcal{O}_{F_1}(f)) = H^1(\mathcal{O}_{\mathbb{P}^1}(1)) = 0\) and \(H^1(\mathcal{O}_{F_1}(s)) \cong H^1(\mathcal{O}_{\mathbb{P}^1}(0, -1)) = 0\). \(\square\)

(2.5) From (2.4.1) we see \(C_{S_1/V_1}|f \cong \mathcal{O}(1,1)\), hence \(S_1 \subset V_1\) is flopped, i.e. there are birational maps

\[
V_1 \xrightarrow{\sigma_2} V_2 \xrightarrow{\sigma_3} V_3,
\]

where \(\sigma_2\) is the blow-up along \(S_1\) with the exceptional divisor \(F_2\), and \(\sigma_3\) is the blow-down of \(F_2\) to the other direction. Let \(S_3 = \sigma_3(F_2)\). Let \(M_2, E_2, T_2, \ldots\) (resp. \(M_3, E_3, T_3, \ldots\)) be the birational transforms on \(V_2\) (resp. \(V_3\)) of \(M_1, E_1, T_1, \ldots\), respectively.

**Lemma.** \(M_3 \cap S_3\) is the \((-1)\)-curve on \(S_3 = \mathbb{F}_1\).

**Proof.** The \((-1)\)-curve \(s_1 = S_1 \cap M_1\) on \(S_1 = \mathbb{F}_1\) by Lemma (2.3)(i); \(C_{s_1/M_1} \cong \mathcal{O}(-1)\). Hence, from (2.4.1), we have a surjection \(C_{s_1/E_1} \cong C_{S_1/V_1}|s_1 \cong \mathcal{O}(0, -1) \to C_{s_1/M_1} \cong \mathcal{O}(-1)\). This defines the closed immersion \(M_2 \cap F_2 = \mathbb{P}[C_{s_1/M_1}] \subset E_2 \cap F_2 = \mathbb{P}[C_{s_1/E_1}]\), which is isomorphic to \(M_3 \cap S_3 \subset S_3\) by \(\sigma_3\). \(\square\)

(2.6) We apply Lemma (0.3)(iii) to \(S = S_1 \cong \mathbb{F}_1\) and \(F = E_1\). Since \(E_1\) intersects \(S_1\) with the \((-1)\)-curve on \(S_1 \cong \mathbb{F}_1\), \(C_{S_1/V_1} \cong \mathcal{O}_{\mathbb{F}_1}(s + f, s)\) implies \(C_{S_3/E_3} \cong \mathcal{O}_{\mathbb{F}_1}(s)\) and there is a split exact sequence

\[
0 \to C_{E_3/V_3}|S_3 \cong \mathcal{O}_{\mathbb{F}_1}(s + f) \to C_{S_3/V_3} \to C_{S_3/E_3} \cong \mathcal{O}_{\mathbb{F}_1}(s) \to 0.
\]

**Lemma.** The exact sequences

\[
\begin{align*}
(2.6.2) & \quad 0 \to C_{L_2/V_2}|T_2 \to C_{T_2/V_2} \to C_{T_2/L_2} \to 0, \\
(2.6.3) & \quad 0 \to C_{L_3/V_3}|T_3 \to C_{T_3/V_3} \to C_{T_3/L_3} \to 0.
\end{align*}
\]

splits with isomorphisms \(C_{L_2/V_2}|T_2 \cong C_{T_2/L_2} \cong \mathcal{O}_{\mathbb{F}_1}(s + f)\) and \(C_{L_3/V_3}|T_3 \cong C_{T_3/L_3} \cong \mathcal{O}_{\mathbb{P}^2}(1)\).

**Proof.** We see \(C_{L_2/V_2}|T_2 \cong C_{L/V}|T \cong \mathcal{O}_{\mathbb{F}_1}(s + f)\), hence we will show \(C_{T_2/V_2} \cong \mathcal{O}_{\mathbb{F}_1}(s + f)\), i.e. \((T_2.f)_{L_2} = -1\) and \((T_2.t_2)_{L_2} = 0\). Let \(Q_2 = \ldots\)
Let \( \mathbb{P}[C_{S_1/V_1}|_{t_1}] \cong \mathbb{P}[C_{t_1/L_1}] \) be the exceptional divisor of \( \sigma_2 : L_2 \to L_1 \). Then \((T_2 \cdot f)_{L_2} = (\sigma_2^* T_1 - Q_2)f = (T_1 \cdot f)_{L_1} - (Q_2 \cdot f)_{L_2} = 0 - 1 = -1\). Since the \((-1)\)-curve \( t_1 \) on \( T_1 \cong \mathbb{F}_1 \) is a fibre on \( S_1 \), \( C_{S_1/V_1}|_{t_1} \cong \mathcal{O}_{S_1}(s + f, s)|_f \cong \mathcal{O}(1,1) \) by (2.4.1). Hence \( t_2 = T_2 \cap Q_2 \subset Q_2 \) is defined by a surjection \( C_{t_1/L_1} \cong C_{S_1/V_1}|_{t_1} \cong \mathcal{O}(1,1) \to C_{t_1/T_1} \cong \mathcal{O}(1) \), so \((Q_2 \cdot t_2) = \mathcal{O}_{Q_2}(-1)|_{t_2} = -1 \) and \((T_2 \cdot t_2)_{L_2} = (\sigma_2^* T_1 - Q_2)t_2 = (T_1 \cdot t_1)_{L_1} - (Q_2 \cdot t_2) = -1 - (-1) = 0 \) since \( C_{T_1/L_1} \cong \mathcal{O}_{T_1}(f) \) by (2.4.2). The exact sequence (2.6.3) follows from \((T_3 \cdot f)_{L_3} = (\sigma_3^* T_3 \cdot f)_{L_2} = (T_2 \cdot f)_{L_2} = -1 \) and \((L_3 \cdot f)_{V_3} = (\sigma_3^* L_3 \cdot f)_{V_2} = (L_2 \cdot f)_{V_2} = -1 \) by (2.6.2). \( \square \)

(2.7) From (2.6.3) we see \( C_{T_3/V_3} \cong \mathcal{O}_{22}(1,1) \), so there are birational maps

\[ V_3 \xrightarrow{\sigma_4} V_4 \xrightarrow{\sigma_5} V_5, \]

where \( \sigma_4 \) is the blow-up along \( T_3 \cong \mathbb{P}^2 \) with the exceptional divisor \( G_4 = \mathbb{P}[C_{T_3/V_3}] \cong \mathbb{P}^2 \times \mathbb{P}^1 \), and \( \sigma_5 \) is the blow-down of \( G_4 \) onto the second factor \( \mathbb{P}^1 \). The birational transform \( S_5 \) on \( V_5 \) of \( S_3 \cong \mathbb{F}_1 \) is the blow-up of \( S_3 \) at the point \( p = S_3 \cap T_3 \). We recall the relative Picard number of \( V_5 \) over \( \text{Spec}(R) \) is equal to two. Let \( f_3 \) be the fibre of \( S_3 = \mathbb{F}_1 \) intersecting at the point \( p = S_3 \cap T_3 \) and let \( f_4 \) (resp. \( f_5 \)) be the birational transform of \( f_3 \) on \( S_4 \) (resp. \( S_5 \)). The two extremal rays of \( V_5 \) over \( X \) are generated by \( f_5 \) and the image \( e_5 = \sigma_5(G_4) \). We see

\[
(-K_{V_4} \cdot f_4)_{V_4} = (\sigma_4^* (-K_{V_3}) - G_4)f_4 = (-K_{V_3} \cdot f_3)_{V_3} - (G_4 \cdot f_4)_{V_4} = (\sigma_5^* (-K_{V_5}) - 2G_4)f_4 = (-K_{V_5} \cdot f_5)_{V_5} - 2(G_4 \cdot f_4)_{V_4}
\]

with \((-K_{V_3} \cdot f_3) = 0\) and \((G_4 \cdot f_4) = 1\), hence \((-K_{V_5} \cdot f_5)_{V_5} = 1\). Let

\[ (2.7.1) \quad \tau_1 : V_5 \to X_1 \]

be the contraction morphism of \( f_5 \).

Lemma. \( (E_5 \cdot f_5)_{V_5} = 0 \).

Proof. From (0.3) we see the exact sequence (2.6.1) induces the split exact sequence

\[ (2.7.2) \quad 0 \to C_{E_5/V_5}|_{S_5} \to C_{S_5/V_5} \to C_{S_5/E_5} \to 0. \]
with isomorphisms $C_{E_5/V_5}|_{S_5} \cong \mathcal{O}_{S_5}(s+f-e)$ and $C_{S_5/E_5} \cong \mathcal{O}_{S_5}(s-e)$. Since $T_3$ is disjoint from $M_3$, Lemma (2.5) means the point $p = S_3 \cap T_3$ is not on the $(-1)$-curve on $S_3 = \mathbb{F}_1$, so that $f_5 = s+f-e$ on $S_5$. Hence we see from (2) that $(E_5.f_5) = -C_{E_5/V_5}|_{f_5} = -(s+f-e)^2 = 0$. □

(2.8) The above Lemma (2.7) implies that any irreducible curve $C$ on $V_5$ with $(E_5.C) = 0$ is contracted by the morphism $(2.7.1)$. We show $E_5$ is covered by such curves. Let $f$ be a fibre of $R \cong \mathbb{F}_1$ on $V$. From Lemma (2.2)(ii), we see $C_{R/V}|_{f} \cong \mathcal{O}(1,0)$, so that $E_{1,f} := \sigma_1^{-1}(f) = \mathbb{P}[C_{R/V}|_{f}]$ is isomorphic to $\mathbb{F}_1$. Let $C$ be a section of $E_{1,f}$ containing the point $E_{1,f} \cap T_1$ (such $C$ exists with 1-parameter family for each fibre $f$ of $R$).

Lemma. $(E_5.C_5)V_5 = 0$ for the birational transform $C_5$ of $C$ on $V_5$.

Proof. For simplicity we use the same letter $C$ for the birational transforms of $C$ on $V_i$ $(1 \leq i \leq 5)$. Since the $(+1)$-curve $C \subset E_{1,f}$ is defined by a surjection $C_{R/V}|_{f} \cong \mathcal{O}(1,0) \to \mathcal{O}(1)$, we see $(E_1.C) = \mathcal{O}_{E_1}(-1)|_C = -1$. Then $(E_3.C) = -1$ because $E_2 = \sigma_2^*E_1 - F_2 = \sigma_2^*E_3 - F_2$. From $E_4 = \sigma_3^*E_3 - G_4 = \sigma_3^*E_5 - G_4$ together with $(G_4.C) = 1$, we see $(E_5.C) = (E_3.C) + (G_4.C) = -1 + 1 = 0$. □

By the above Lemma, there are 1-parameter family of the curves $C$ with $(E_5.C) = 0$ for each point on $e = \sigma_5(G_4)$, so the image of the morphism $(2.7.1)$ is 2-dimensional. This means $\tau_1 : V_5 \to X_1$ is a standard $\mathbb{P}^2$-bundle and the structure morphism $X_1 \to \text{Spec}(R)$ is the blow-up at the origin. The statement (II) of Theorem is proved.

(2.9) Let $X$ be a smooth algebraic surface and let $Y \to X$ be the blow up at a point of $X$ with the exceptional line $e$. Let $\tau_1 : W \to Y$ be a standard $\mathbb{P}^2$-bundle with the non-smooth locus intersecting $e$ transversely at one point $p_0 = \Delta \cap e$. If we find a smooth subvariety $S_5$ of $\tau_1(e)$ such that

(2.9.1) $S_5$ is a $\mathbb{P}^1$-bundle over $e$ away from the point $p_0 = e \cap \Delta$,
(2.9.2) there is a section $e_5$ on $S_5$ over $e$ with $(e_5^2)_{S_5} = -1$,

then we obtain a standard $\mathbb{P}^2$-bundle $V$ over $X$ by applying the inverse of the birational map (II) described above. We see the existence of such subvarieties $e_5 \subset S_5$ by the following argument. As in (2.1), $\tau_1^{-1}(e)$ is obtained from a $\mathbb{P}^2$-bundle $\tau_2 : P \to e$ by twice blow-ups

$$
\tau_1^{-1}(e) \xrightarrow{\psi_3} P_1 \xrightarrow{\psi_1} P,
$$

— 28 —
where $\epsilon_1$ is the blow-up of $P$ along a line $l$ in $\tau_2^{-1}(p_0) \cong \mathbb{P}^2$, and $\epsilon_2$ is the blow-up of $P_1$ along a fibre $f$ of the exceptional divisor $\epsilon_1^{-1}(l)$ of $\epsilon_1$. We write $P = \mathbb{P}[\mathcal{E}]$ with a rank three vector bundle $\mathcal{E} \cong \mathcal{O}(0, a, b)$ on $e \cong \mathbb{P}^1$. Let $\phi : \mathcal{E}\mathcal{O}(c, c+1)$ be a surjection for an integer $c \in \mathbb{Z}$, and $e_0 = \mathbb{P}[\mathcal{O}(c)] \subset S_0 = \mathbb{P}[\mathcal{O}(c, c+1)]$ be the corresponding subvarieties of $P = \mathbb{P}[\mathcal{E}]$. We choose $\phi : \mathcal{E} \rightarrow \mathcal{O}(c, c+1)$ such that

(2.9.3) the line $S_0 \cap \tau_2^{-1}(p_0)$ is not equal to the center $l$ of $\epsilon_1$,
(2.9.4) $e_0 \cap \tau_1^{-1}(p_0), S_0 \cap l, \epsilon_1(f)$ are distinct three points on $\tau_2^{-1}(p_0)$.

Then we see the proper transforms in $\tau_1^{-1}(e)$ of $S_0$ and $e_0$ satify (2.9.1) and (2.9.2).

3. The birational map (III)

(3.1) In this section we consider the birational map (III). Let $\tau : V \rightarrow X$ be a standard $\mathbb{P}^2$-bundle over a smooth algebraic surface $X$ and let $C \subset X$ be a curve intersecting the non-smooth locus $\Delta$ of $\tau$ transversely at one smooth point $p_0$ of $\Delta$. Let $C_0 \subset V$ be a curve which is isomorphic to $C$ by $\tau$. Let $\tau^{-1}(p_0) = R \cup S \cup T$ with $R, S, T$ isomorphic to $\mathbb{F}^1$ and assume $s = S \cap R$ is the $(-1)$-curve on $S$.

Lemma. There is a split exact sequence

$$0 \rightarrow C_{\tau^{-1}(C)/V}|_S \cong \mathcal{O}_{\mathbb{F}^1} \rightarrow C_{S/V} \rightarrow C_{S/\tau^{-1}(C)} \cong \mathcal{O}_{\mathbb{F}^1}(s + f) \rightarrow 0.$$

Proof. $C_{\tau^{-1}(C)/V}|_S \cong \tau^*\mathcal{O}_X(-C)|_S \cong \mathcal{O}_{\mathbb{F}^1}$ is clear. Since $R + S + T \equiv 0$ on $\tau^{-1}(C)$, we see $(S.f)_{\tau^{-1}(C)} = -(R.f) - (T.f) = -1 - 0 = -1,$ and $(S.s)_{\tau^{-1}(C)} = 0$ because $s$ is a fibre of $R$. Hence $C_{S/\tau^{-1}(C)} \cong \mathcal{O}_{\mathbb{F}^1}(s + f)$. 

(3.2) Let $\sigma_1 : V_1 \rightarrow V$ be the blow-up along $C_0$ with the exceptional divisor $E_1$ and let $R_1, S_1, T_1, D_1$ be the proper transforms of $R, S, T, \tau^{-1}(C)$, respectively. The restriction to $S_1$ of $\sigma_1$ is the blow-up of $S$ at the point $p = \Delta \cap C_0$. Let

$$e_1 = S_1 \cap E_1$$

be the exceptional line of $\sigma_1 : S_1 \rightarrow S \cong \mathbb{F}^1$. 


Lemma. The exact sequence

(3.2.1) \[ 0 \to C_{D_1/V_1\mid s_1} \to C_{S_1/V_1} \to C_{S_1/D_1} \to 0 \]
splits with isomorphisms \( C_{D_1/V_1\mid s_1} \cong \mathcal{O}_{S_1}(e_1) \) and \( C_{S_1/D_1} \cong \mathcal{O}_{S_1}(s_1 + f_1) \).

Proof. \( C_{D_1/V_1\mid s_1} \cong \mathcal{O}_{S_1}(e_1) \) follows from

\[
\begin{align*}
(D_1.s_1)_V &= (\sigma_1^*D - E_1)s_1 = (D.s)_V - (E_1.s)_V = 0 - 0 = 0, \\
(D_1.f_1)_V &= (\sigma_1^*D - E_1)f_1 = (D.f)_V - (E_1.f)_V = 0 - 0 = 0, \\
(D_1.e_1)_V &= (\sigma_1^*D - E_1)e_1 = -(E_1.e_1)_V = 0.
\end{align*}
\]

Similarly, \( C_{S_1/D_1} \cong \mathcal{O}_{S_1}(s_1 + f_1) \) follows from the equalities \( (S_1.s_1)_D_1 = (\sigma_1^*S.s_1) = (S.s)_{r-1(C)} = 0, (S_1.f_1)_V = (\sigma_1^*S.f_1) = (S.f)_{r-1(C)} = -1 \) and \( (S_1.e_1)_D_1 = (\sigma_1^*S.e_1) = 0. \) Since \( H^1(\mathcal{O}_{S_1}(-s_1 - f_1 + e_1)) = H^1(\mathcal{O}_{S_1}(-s_1 - 2f_1)) \) (Serre duality) = \( H^1(\mathcal{O}_{F_1}(-s - 2f)) = 0 \), the exact sequence (3.2.1) splits.

(3.3) There is a blow-down \( S_1 \to \mathbb{F}_0 \) with the exceptional line \( e = f_1 - e_1 \). We take

(3.3.1) \[ s = f_1, \quad f = s_1 + f_1 - e_1 \]
as the section and the fibre of \( S_1 \) induced from those of \( \mathbb{F}_0 \). Then \( s - e = e_1 \) and \( s + f - e = s_1 + f_1 \), so the exact sequence (3.2.1) is equal to

(3.3.2) \[ 0 \to C_{D_1/V_1\mid s_1} \to C_{S_1/V_1} \to C_{S_1/D_1} \to 0. \]

with isomorphisms \( C_{D_1/V_1\mid s_1} \cong \mathcal{O}_{S_1}(s - e) \) and \( C_{S_1/D_1} \cong \mathcal{O}_{S_1}(s + f - e) \). Hence \( C_{S_1/V_1\mid e} \cong \mathcal{O}(1, 1) \), so there are birational maps

\[ V_1 \xrightarrow{\sigma_2} V_2 \xrightarrow{\sigma_3} V_3, \]

where \( \sigma_2 \) is the blow-up of \( e \) with the exceptional divisor \( B_3 \cong \mathbb{P}^1 \times \mathbb{P}^2 \), and \( \sigma_3 \) is the blow-down of \( B_3 \) to the other direction. Let \( D_3, S_3, \ldots \) be the proper transforms on \( V_3 \) of \( D_1, S_1, \ldots \), respectively. We see from (0.3) that the exact sequence (3.3.2) induces

(3.3.3) \[ 0 \to C_{D_3/V_3\mid s_3} \to C_{S_3/V_3} \to C_{S_3/D_3} \to 0. \]
with isomorphisms $\mathcal{C}_{D_3/V_3}\vert_{S_3} \cong \mathcal{O}_{\mathbb{F}_0}(s)$ and $\mathcal{C}_{S_3/D_3} \cong \mathcal{O}_{\mathbb{F}_0}(s + f)$. Therefore $\mathcal{C}_{S_3/V_3}\vert_f \cong \mathcal{O}(1,1)$, so that $S_3 \subset V_3$ is flopped, i.e. there are birational maps

$$V_3 \xrightarrow{\sigma_4} V_4 \xrightarrow{\sigma_5} V_5,$$

where $\sigma_4$ is the blow-up of $S_3$ with the exceptional divisor $F_4$, and $\sigma_5$ is the blow-down of $F_4$ to the other direction. The flopped surface $S_5$ on $V_5$ is isomorphic to $\mathbb{P}[\mathcal{C}_{S_3/V_3}\vert_v] \cong \mathbb{P}[\mathcal{O}(-1,0)] = \mathbb{F}_1$ by (3.3.3), and satisfies $\mathcal{C}_{S_5/V_5} \cong \mathcal{O}_{\mathbb{F}_1}(s + f, s + f)$ by Lemma (0.3)(i).

(3.4) The extremal rays on $V_5$ over $X$. One is generated by the fibre $f_5$ of the flopped surface $S_5 \cong \mathbb{F}_1$, and the other is the birational transform $l_5$ of the line $l$ in $\tau^{-1}(p) \cong \mathbb{P}^2$ with $l \cap \mathcal{C}_0$ non-empty for a point $p \in C - (C \cap \Delta)$. We see $(-K_{V_5}.f_5) = 0$, $(-K_{V_5}.l_5) = 1$, $(E_5.f_5) = -1$ and $(E_5.l_5) = 1$. The birational transform $D_5$ on $V_5$ of $\tau^{-1}(C)$ has a $\mathbb{P}^1$-bundle structure over the surface $D_5 \cap E_5$ with fibre $l_5$, hence the contraction morphism of $l_5$

$$\sigma_6 : V_5 \to V_6,$$

is the blow-up of $V_6$ along the surface $\sigma_6(D_5)$. The structure morphism $V_6 \to X$ defines the standard $\mathbb{P}^2$-bundle over $X$.

4. Appendix

(4.1) The standard $\mathbb{P}^2$-bundle $V$ over the local ring $R$ of a singular point of $\Delta$ constructed from the $R$-algebra (0.1.3), is described as follows.

**lemma.** [M,(2.4)] (i) $V$ is covered by three open sets $U_3, U_{11}, U_{12}$, which are isomorphic to the affine space $\mathbb{A}^4_k$ of dimension four with affine coordinates $(f, x_1, x_2, x_3)$, $(g, y_8, y_5, y_3)$, $(w_{12}, w_2, w_5, w_{11})$, respectively, such that the transition functions are given by

(a) $U_{11}$ to $U_{12}$:

- $f = y_8^3 + gy_5^3 + g^2y_3^3 - 3gy_3y_5y_3$,
- $x_1 = (y_3y_8^2 - \omega y_5^2y_8 - \omega^2 gy_3^2y_5)/y_{12}$,
- $x_2 = (y_5^2 + y_3y_8)/y_{12}$,
- $x_3 = y_3/y_{12}$,
- with $y_{12} = \omega^2y_5^3 + \omega gy_3^3 + (\omega^2 - \omega)y_3y_5y_8$,

(b) $U_{12}$ to $U_{11}$:

- $g = x_1^3 + fx_2^3 + f^2x_3^3 - 3fx_1x_2x_3$,
- $y_8 = (-x_1x_2^2 - \omega x_1x_3 - \omega^2 fx_2x_3^2)/x_{11}$.
\[ y_5 = \omega^2(x_2^2 - x_1 x_3)/x_{11}, \quad y_3 = x_3/x_{11}, \]

with \[ x_{11} = \omega x_2^3 - \omega^2 f x_3^3 + (\omega^2 - \omega)x_1 x_2 x_3, \]

(c) \( U_3 \) to \( U_{12} \):
\[
(f = w_2^3 - \omega w_{11} w_{12}^2 + (1 - \omega)w_2 w_5 w_{12} \\
= w_2 (w_2^2 - \omega w_5 w_{11}) + w_{12} (w_2 w_5 - \omega w_{11} w_{12}), \\
x_1 = (w_2^2 - \omega w_5 w_{12})/w_{12}, \\
x_2 = w_2/w_{12}, \quad x_3 = 1/w_{12},
\]

(d) \( U_{12} \) to \( U_3 \):
\[
w_{12} = 1/x_3, \quad w_2 = x_2/x_3, \\
w_5 = \omega^2(x_2^2 - x_1 x_3)/x_3, \quad w_{11} = x_{11}/x_3,
\]

(e) \( U_3 \) to \( U_{11} \):
\[
-g = w_5^3 - \omega^2 w_{11}^2 w_{12} + (\omega - 1)w_2 w_5 w_{11}, \\
= w_5 (w_5^2 - w_2 w_{11}) + \omega w_{11} (w_2 w_5 - \omega w_{11} w_{12}), \\
y_8 = (w_2 w_{11} - w_5^2)/w_{11}, \\
y_5 = w_5/w_{11}, \quad y_3 = 1/w_{11},
\]

(f) \( U_{11} \) to \( U_3 \):
\[
w_{12} = y_{12}/y_3, \quad w_2 = (y_5^2 + y_3 y_8)/y_3, \\
w_5 = y_5/y_3, \quad w_{11} = 1/y_3,
\]

(ii) The projection \( \tau : V \to \text{Spec}(R) \) is given by
\[
\tau(f, x_1, x_2, x_3) = (f, x_1^3 + f x_2^3 + f^3 x_3^3 - 3 f x_1 x_2 x_3) \quad \text{on } U_{12}, \\
\tau(g, y_8, y_5, y_3) = (y_8^3 + g y_5^3 - g^2 y_3^3 + 3 g y_3 y_5 y_8, g) \quad \text{on } U_{11}, \\
\tau(w_{12}, w_2, w_5, w_{11}) = (w_5^3 - \omega w_{11} w_{12}^2 + (1 - \omega)w_2 w_5 w_{12}, \\
- w_5^3 + \omega^2 w_{11} w_{12} + (1 - \omega)w_2 w_5 w_{11}) \quad \text{on } U_3,
\]

(iii) The central fibre \( \tau^{-1}(p) \) with reduced structure is defined by the ideal
\[
(f, x_1) \quad \text{on } U_{12}, \quad (g, y_8) \quad \text{on } U_{11}, \\
(\omega w_{12} w_5 - w_2, \omega w_{11} w_{12} - w_2 w_5, w_2 w_{11} - w_5^2) \quad \text{on } U_3,
\]
and the vertex of \( \tau^{-1}(p)_{\text{red}} \) is the origin of \( U_3 \cong A^4_k \).

(4.2) (Proof of Lemma(1.2)) By Lemma(4.1)(iii) we assume the fibre \( l \) is equal to \( l = \{ \lambda(1, a, \omega^2 a^2, \omega a^3) | \lambda \in k \} \) on \( U_3 \cong A^4, (w_{12}, w_2, w_5, w_{11}) \), for a constant \( a \in k \). Let
\[
w'_2 = w_2 - a w_{12}, \quad w'_5 = w_5 - \omega^2 a^2 w_{12}, \quad w'_{11} = w_{11} - \omega a^3 w_{12},
\]

— 32 —
Then we see (where \(\equiv\) means modulo the ideal \((w'_2, w'_5, w'_{11})^2)\)

\[
\begin{align*}
  w^2_2 - \omega w_5 w_{12} &= (w'_2 + aw_{12})^2 - \omega (w'_5 + \omega^2 a^2 w_{12}) w_{12} \\
  &\equiv (2aw'_2 - \omega w'_5) w_{12},
\end{align*}
\]

\[
\begin{align*}
  w_2 w_5 - w_{12} w_{11} &= (w'_2 + aw_{12})(w'_5 + \omega^2 a^2 w_{12}) - \omega (w'_{11} + \omega a^3 w_{12}) w_{12} \\
  &\equiv \omega^2 a^2 w'_2 + aw'_5 - \omega w'_{11}.
\end{align*}
\]

Hence, by Lemma(4.1)(i)(c),

\[
\begin{align*}
  f &\equiv (w'_2 + aw_{12})(2aw'_2 - \omega w'_5) w_{12} + w_{12}^2 (\omega^2 a^2 w'_2 + aw'_5 - \omega w'_{11}) \\
  &\equiv w_{12}^2 \{(1 - \omega) a^2 w'_2 + a(1 - \omega) w'_5 - \omega w'_{11}\}
\end{align*}
\]

\[
\begin{align*}
  x_1 &= (w^2_2 - \omega w'_5 w'_{11})/w_{12} + (2aw'_2 - \omega w'_5), \\
  x_2 &= (w'_2 + aw_{12})/w_{12}, \\
  x_3 &= 1/w_{12}.
\end{align*}
\]

Therefore, on the fibre \(l = \{w'_2 = w'_5 = w'_{11} = 0\}\), \(t(df, dx_1, d(x_2 - a), dx_3)\) is equal to

\[
\begin{pmatrix}
  -\omega w_{12}^2 & a(1 - \omega) w_{12}^2 & (1 - \omega) w_{12}^2 & 0 \\
  0 & -\omega & 2a & 0 \\
  0 & 0 & 1/w_{12} & 0 \\
  0 & 0 & 0 & -1/w_{12}^2
\end{pmatrix}
\begin{pmatrix}
  dw'_{11} \\
  dw'_5 \\
  dw'_2 \\
  dw_{12}
\end{pmatrix}.
\]

Then \(C_{\sigma_1(l)/V} \cong \mathcal{O}(2, 0, -1)\) follows from the above transition matrix of (4.2.1).

(4.3) Lastly we explain briefly the three birational maps (I)-(III) of standard conic bundles (cf.[Sa]) corresponding to those of \(\mathbb{P}^2\)-bundles treating in this paper. Let \(\tau : V \to X\) be a standard conic bundle over a smooth algebraic surface \(X\).

(I) Let \(p\) be a singular point of the discriminant locus \(\Delta\) of \(V\) and \(\sigma : X_1 \to X\) be the blow up at \(p\). The reduced fibre \(l = \tau^{-1}(p)_{\text{red}} \subset V\) is isomorphic to \(\mathbb{P}^1\) and the conormal bundle \(C_{l/V} \cong \mathcal{O}_{\mathbb{P}^1}(2, -1)\). Let \(\sigma_1 : V_1 \to V\) be the blow-up along \(l\) and let \(s\) be the \((-3)\)-curve on the exceptional divisor \(E = \mathbb{P}[C_{l/V}] \cong \mathbb{F}_3\). Then \(C_{s/V_1} \cong \mathcal{O}(1, 1)\), so \(s \subset V_1\) is flopped to \(s^+ \subset W\). Now \(W\) has a conic bundle structure over \(X_1\) with the non-smooth locus equal to the union of the exceptional
divisor $e$ and the proper transform $\Delta'$ of $\Delta$. The birational map (I) is factored by

$$(4.3.1) \quad V \overset{\varphi_1}{\twoheadrightarrow} V_1 \rightarrow W.$$ 

The flopped curve $s^+ \subset W$ is the closure of the singular locus of $\tau^{-1}(e - \Delta')$.

(II) Let $p$ be a smooth point of the discriminant locus $\Delta$ of $V$ and $\sigma : X_1 \to X$ be the blow-up at $p$. The fibre $\tau^{-1}(p) = s \cup m$ is a union of two distinct lines $s$ and $m$. Let $\sigma_1 : V_1 \to V$ be the blow-up along $m$ and let $s_1 \subset V_1$ be the proper transform of $s$. Then $\mathcal{C}_{s_1/V_1} \cong \mathcal{O}(1,1)$, so $s_1 \subset V_1$ is flopped to $s^+ \subset W$. Now $W$ has a conic bundle structure over $X_1$ with the non-smooth locus equal to the proper transform of $\Delta$. The birational map (II) is factored as in (4.3.1). The flopped curve $s^+$ is a section over the exceptional curve $e$ on $X_1$ with $\mathcal{C}_{s^+/\tau^{-1}(e)} \cong \mathcal{O}(1)$.

(III) Let $C \subset X$ be a smooth curve intersecting transversely at one smooth point $p$ of $\Delta$. Let $C_0 \subset V$ be a curve which is isomorphic to $C$ by $\tau$. The fibre $\tau^{-1}(p) = s \cup m$ is the union of two lines $s$ and $m$, and we assume $C_0$ intersects $s$. Let $\sigma_1 : V_1 \to V$ be the blow-up along $C_0$ and let $s_1 \subset V_1$ be the proper transform of $s$. Then $\mathcal{C}_{s_1/V_1} \cong \mathcal{O}(1,1)$, so $s_1 \subset V_1$ is flopped to $s^+ \subset V_2$. The birational transform $F \subset V_2$ of $\tau^{-1}(C)$ is a $\mathbb{P}^1$-bundle over $C$ and its fibre $f$ is an extremal rational curve. Let $\sigma_2 : V_2 \to W$ be the contraction of $f$. Then $W$ has a conic bundle structure over $X$ with the same non-smooth locus $\Delta$ of $V$. The birational map (III) is factored by

$$V \overset{\varphi_1}{\twoheadrightarrow} V_1 \rightarrow V_2 \overset{\varphi_2}{\twoheadrightarrow} W.$$ 

If $C$ is isomorphic to $\mathbb{P}^1$ with $\mathcal{C}_{C/X} \cong \mathcal{O}(a)$ and $\mathcal{C}_{C_0/\tau^{-1}(C)} \cong \mathcal{O}(b)$ for integers $a, b \in \mathbb{Z}$, then $\mathcal{C}_{\sigma_2(F)/\tau^{-1}(C)} \cong \mathcal{O}(a-b+1)$. In particular, if $C$ is a $(-1)$-curve with $\mathcal{C}_{C_0/\tau^{-1}(C)} \cong \mathcal{O}(b)$, then $\mathcal{C}_{\sigma_2(F)/\tau^{-1}(C)} \cong \mathcal{O}(-b+2)$.

REFERENCES


Department of Mathematical Sciences  
College of Science  
University of Ryukyus  
Nishihara Okinawa 903-01  
Japan