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<th>Parameter and Reliability Estimation for a Bivariate Exponential Distribution</th>
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<tr>
<td>Author(s)</td>
<td>Nakao, Zensho; Liu, Zeng-Zhong; Kinjo, Masaya</td>
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Parameter and Reliability Estimation for a Bivariate Exponential Distribution†

Zensho NAKAO*, Zeng-Zhong LIU**, and Masaya KINJO*

Abstract

Parameter and reliability estimators for a bivariate exponential distribution are derived by the moment and the maximum likelihood methods.

Key Words: Bivariate exponential distribution, Maximum likelihood estimation, Moment estimation, Parameters, Reliability function

1. Introduction

System elements are connected in series and/or in parallel. For example, in electric circuits, two resistors may be connected in series or in parallel; in electric power distribution, two generators may be used in parallel to supply energy; in multi-computer systems (space shuttles, for example), three or more computers may be connected (redundantly) in parallel for reliability.

Suppose that such a system consists of two components in series or in parallel, and is shocked by three Poisson random interferences from outside: one shocking the first element, the second striking the second, and the third hitting the two components simultaneously. Then the survival probability function for the system can be shown to be given by a bivariate exponential distribution (BVE) with parameters \( \lambda_{10} \), \( \lambda_{10} \), and \( \lambda_{11} \): \( F(x, y) = P(X > x, Y > y) = \exp\left[ -\lambda_{10} x - \lambda_{10} y - \lambda_{11} \max(x, y) \right] \), where \( X, Y \) denote the lifelength of the two components, \( \lambda_{10} > 0, \lambda_{10} > 0, \lambda_{11} > 0, x, y \geq 0 \).

We will present a set of estimators for the parameters and the reliability function of such systems by the method of moments and the maximum likelihood method. Bayesian and empirical Bayesian methods for estimating the parameters and the reliability function of a parallel BVE system where past data are available are given in [4, 5].

2. Derivation of a BVE

Let two independent Poisson processes \( Z_1(t; \lambda_{10}) \), \( Z_2(t; \lambda_{10}) \) shock the components, numbered 1, 2, respectively, and let a third Poisson process \( Z_{11}(t; \lambda_{11}) \) shock the components 1, 2 simultaneously. Let \( X, Y \) denote the lifelength of the 1, 2 components, and define the survival probability function by \( F(x, y) = P(X > x, Y > y) \). Then \( F(x, y) = P[Z_1(x; \lambda_{10}) = 0, Z_2(y; \lambda_{10}) = 0, Z_{11}(\max(x, y); \lambda_{11}) = 0] = \exp(-\lambda_{10} x) \cdot \exp(-\lambda_{10} y) \cdot \exp(-\lambda_{11} \max(x, y)), \) where \( \lambda_{10} > 0, \lambda_{10} > 0, \lambda_{11} > 0, x, y \geq 0 \).

Recall that a Poisson random process \( Z(t; \lambda) \) is given by

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Parameter and Reliability Estimation for a Bivariate Exponential Distribution

3. Parameter and reliability estimation for a BVE

a. Moment estimation for a BVE

It is known [3] that

\[ E[X] = \frac{1}{\lambda_{01} + \lambda_{11}} \, , \quad E[Y] = \frac{1}{\lambda_{10} + \lambda_{11}} \, , \quad E[XY] = \frac{l}{\lambda} \left( \frac{1}{\lambda_{01} + \lambda_{11}} + \frac{1}{\lambda_{10} + \lambda_{11}} \right) \; ; \]

(where \( \lambda = \lambda_{01} + \lambda_{10} + \lambda_{11} \), and \( E[ ] \) denotes the expected value).

Using the method of moments, we obtain a system of equations:

\[ \sum_{i=1}^{n} \frac{x_i}{n} = \frac{1}{\lambda_{01} + \lambda_{11}} \, , \quad \sum_{i=1}^{n} \frac{y_i}{n} = \frac{1}{\lambda_{10} + \lambda_{11}} \, , \quad \sum_{i=1}^{n} \frac{x_i y_i}{n} = \frac{l}{\lambda} \left( \frac{1}{\lambda_{01} + \lambda_{11}} + \frac{1}{\lambda_{10} + \lambda_{11}} \right) \; ; \] (n: sample size).

Solving this system for the parameters, we get the estimators:

\[ \hat{\lambda}_{01} = \frac{n}{\sum_{i=1}^{n} x_i y_i} - \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} y_i} \, , \quad \hat{\lambda}_{10} = \frac{n}{\sum_{i=1}^{n} x_i y_i} - \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} y_i} \, , \quad \hat{\lambda}_{11} = \frac{n}{\sum_{i=1}^{n} x_i y_i} - \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} y_i} \, . \]

Our method is slightly different from the moment method in [2] in that their third equation differs from ours given above, and that \(|\{i : x_i = y_i\}| = 0\) (\(|\cdot\) denotes the cardinality of a set), then \(\lambda_{11} = 0\) in [2], which is a difficulty in that method. For a parallel system, the system is in an unfailed state at time \(t\), if at least one components survive at the time \(t\). Hence the reliability of the system is given by

\[ R(t) = P(X > t \text{ or } Y > t) = P(X > t) + P(Y > t) - P(X > t, Y > t) = F(t, 0) + F(0, t) - F(t, t) = \exp[-(\hat{\lambda}_{01} + \hat{\lambda}_{11}) t] + \exp[-(\hat{\lambda}_{10} + \hat{\lambda}_{11}) t] - \exp[-(\hat{\lambda}_{01} + \hat{\lambda}_{10}) t] . \]

Thus the corresponding estimator \( \hat{R}(t) \) for the reliability can be obtained by replacing \( \lambda \)'s with their estimators \( \hat{\lambda} \)'s:

\[ \hat{R}(t) = \exp[-(\hat{\lambda}_{01} + \hat{\lambda}_{11}) t] + \exp[-(\hat{\lambda}_{10} + \hat{\lambda}_{11}) t] + \exp(-\hat{\lambda}_{01} t) \]  

\( (r \geq 0) \).

b. Maximum likelihood estimation for a BVE

For parallel systems, the maximum likelihood estimation for a BVE was made by F. Prochan and P. Sullo [6]; here we will derive maximum likelihood estimators for a BVE in series, which turns out to be much simpler than for parallel systems. Let \((X, Y) \sim \text{BVE}(\lambda_{01}, \lambda_{10}, \lambda_{11}), \) and \(S = \{(0,0), (0,1), (1,1)\} \) .

For \( s \in S \), define a characteristic function \( V_s \) as follows [1] :

\[ V_{(a,b)} = \begin{cases} 1 & \text{if } X < Y, \\ 0 & \text{if } X \geq Y. \end{cases} \quad V_{(b,a)} = \begin{cases} 1 & \text{if } X > Y, \\ 0 & \text{if } X \leq Y. \end{cases} \quad V_{(a,a)} = \begin{cases} 1 & \text{if } X = Y, \\ 0 & \text{if } X \neq Y. \end{cases} \]

Thus, for every sample \((X_i, Y_i), i = 1, \ldots, n,\) we have \( V_s \), defined, \( i = 1, \ldots, n. \) Let \( N_{i} = \sum_{s \in S} V_{s} \), and let \( U_i = \min(X_i, Y_i), i.e., \) the life time of the series system \( i = 1, \ldots, n. \) Now, for a series system, the entire system fails when one unit fails. So the only information available is on the life time of the system, \( U_i, i = 1, \ldots, n, \) and on which unit has the longer life time, \( V_{s,i}, i = 1, \ldots, n. \) It is obvious that \( V_{s,i}, i = 1, \ldots, n, \) are independent of \( U_j, j = 1, \ldots, n \) [1], and moreover, \(( N_{(0,0)}, N_{(0,1)}, N_{(1,1)} )\) has a trinomial distribution with probabilities \( \frac{\lambda_{01}}{\lambda}, \frac{\lambda_{10}}{\lambda}, \frac{\lambda_{11}}{\lambda} \), respectively. So the joint probability is given (where \( n = N_{(0,0)} + N_{(0,1)} + N_{(1,1)} \) ) by

\[ P[N_{(0,0)} = n_{(0,0)}, N_{(0,1)} = n_{(0,1)}, N_{(1,1)} = n_{(1,1)} | U_i > t, i = 1, \ldots, n ] = \begin{cases} 1 & \text{if } N_{(0,0)} = n_{(0,0)}, N_{(0,1)} = n_{(0,1)}, N_{(1,1)} = n_{(1,1)} \end{cases} P[U_i > t, i = 1, \ldots, n] \]

\[ = \frac{n_{(0,0)}! \cdot n_{(0,1)}! \cdot n_{(1,1)}!}{n!} \cdot \frac{\lambda_{01}^{n_{(0,0)}} \cdot \lambda_{10}^{n_{(0,1)}} \cdot \lambda_{11}^{n_{(1,1)}} \cdot \exp(-\lambda \sum_{i=1}^{n} U_i)}{\cdot} \]

To derive maximum likelihood estimators, we take the logarithm of the likelihood function above, and set its gradient equal to zero:
Solving the resulting system of equations, we get the following estimators:

\[ \hat{\lambda}_{01} = \frac{n(0 , 1)}{\sum_{i=1}^{n} U_i}, \quad \hat{\lambda}_{10} = \frac{n(1 , 0)}{\sum_{i=1}^{n} U_i}, \quad \hat{\lambda}_{11} = \frac{n(1 , 1)}{\sum_{i=1}^{n} U_i} \]

For a series system, the system is in a failed state at time \( t \) if at least one component fails at the time \( t \). Hence the reliability of the system is given by

\[ R(t) = P(X > t, Y > t) = \exp(-\lambda t). \]

Thus the corresponding estimator \( \hat{R}(t) \) for the reliability becomes

\[ \hat{R}(t) = \exp(-\hat{\lambda} t) = \exp\left(\frac{-n}{\sum_{i=1}^{n} U_i} t\right), \quad (t \geq 0). \]

4. Simulation on parameter and reliability estimation

a. Methods

We give some detail for computer simulation: First, \( RND(p) \) \( [p \in (0, 1)] \) is used to produce random values in the unit interval \( (0, 1) \). Secondly, we let \( U - E(\lambda_{01}), V - E(\lambda_{10}), W - E(\lambda_{11}), (E(\lambda) : \lambda) : \) an exponential distribution with parameter \( \lambda \), and due to the fact that \( 1 - \exp(-\lambda_{01} U) - R(0 , l), 1 - \exp(-\lambda_{10} V) - R(0 , l), 1 - \exp(-\lambda_{11} W) - R(0 , l) \) (where \( R(0 , l) \) is a uniform distribution in the interval \( (0 , l) \)), we get samples (of size \( n \)) \( U_1, \ldots, U_n; V_1, \ldots, V_n; W_1, \ldots, W_n \). Thirdly, we obtain independent random variables \( U, V, W, X = \min(U, W) \), and \( Y = \min(V, W) \), and get sample values of \( (X, Y) \) which is known to have a BVE distribution [3].

b. Results and analysis

Results of simulation are shown for moment estimation of parallel systems and for maximum likelihood estimation of series systems in Tables 1 and 2, respectively. We used the mean squared errors (MSE) for evaluation of the estimators, where

\[ \text{MSE} = \frac{1}{n} \sum_{i=1}^{n} (\text{estimator} - \text{true value})^2. \]

Table 1

<table>
<thead>
<tr>
<th>N</th>
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<th>( \lambda_{10} )</th>
<th>( \lambda_{11} )</th>
<th>( R )</th>
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<td>0.0035</td>
<td>0.0010</td>
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<table>
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Table 2

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(N: Sample sizes, MSE: Mean Squared errors, \( t = 1 \))
Table 2  Parameter and reliability estimators of a BVE
(Maximum likelihood method)

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<th>$\lambda_{11}$</th>
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<table>
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<th>$\lambda_{10}$</th>
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(N: Sample sizes, MSE: Mean Squared errors, $t = 1$)

In Table 1, $0.0002 \leq \text{MSE (reliability)} \leq 0.0014; 0.0011 \leq \text{MSE (}$\lambda_{01}$\text{)} \leq 0.0057; 0.0015 \leq \text{MSE (}$\lambda_{10}$\text{)} \leq 0.0035; 0.0007 \leq \text{MSE (}$\lambda_{11}$\text{)} \leq 0.0037; while in Table 2, 0.003 \leq \text{MSE (reliability)} \leq 0.0005; 0.0001 \leq \text{MSE (}$\lambda_{01}$\text{)} \leq 0.0015; 0.0005 \leq \text{MSE (}$\lambda_{10}$\text{)} \leq 0.0008; 0.0005 \leq \text{MSE (}$\lambda_{11}$\text{)} \leq 0.0008

We see that, from Tables 1 and 2, the MSE's for both the moment and the maximum likelihood estimation are small enough as expected of the methods, and that, from Table 3, the MSE's for the maximum likelihood estimators are consistently smaller than those for the moment estimators.

5. Conclusions

We obtained parameters and reliability estimators for a BVE where the system components are connected in series or in parallel; used the MSE's for evaluating the goodness of estimation and found that the MSE's are small enough as expected of the methods; in general, the maximum likelihood estimation suits better than the moment estimation for the parameters and the reliability function of a BVE; thus, for parallel systems, the estimators obtained in [6] and, for series systems, our estimators can be adopted.

There remain problems of hypothesis testing on the parameters of a BVE, which will be the subjects for our next investigation.

References

