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A Non-parametric Test for Equivalence of Marginal Distributions of a Bivariate Exponential Distribution

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Abstract

An asymptotically non-parametric method was used to test the equivalence of the marginal distributions of a bivariate exponential distribution and to compute Pitman's relative efficiency.

1. Introduction

We will use an asymptotically non-parametric method due to Raviv[5] to test the equivalence of the marginal distributions of a bivariate exponential distribution (BVED) [3, 4]. Raviv applied the method to a general continuous distribution and compared the method to the paired-t-test as well; the result was found to be satisfactory. We will use the method (called the R-test) to test the equivalence of the marginal distributions of a BV-ED and to calculate Pitman's relative efficiency. We will see that the minimum Pitman's relative efficiency is 3. By simulation, we will also find that the difference between the power of R-test and that of the t-test is small.

2. R-test and its applications

Assume $\binom{x_1}{y_1}, \cdots, \binom{x_n}{y_n} \sim \text{BVED}(\lambda_1, \lambda_2, \lambda_0)$ and note that the marginal distributions are given by

$$F(x) = 1-e^{-\nu_1 x},\ G(y) = 1-e^{-\nu_2 y},\ x, y \geqq 0 \quad \text{where} \quad \nu_1 = \lambda_1 + \lambda_0, \nu_2 = \lambda_2 + \lambda_0.$$

We are going to test the following:

$$H_0 : \nu_1 = \nu_2 \longleftrightarrow H_1 : \nu_1 < \nu_2 .$$

Define $Z_{ij} = \begin{cases} 1 & x_i > y_j, \\ 0 & \text{otherwise}, \end{cases} \qquad i, j = 1, 2 \cdots n;$

$$R = \sum_{i \neq j} Z_{ij} = \sum_i \sum_j Z_{ij} - \sum_i Z_{ii} = U - S,$$

$$T_i = \sum_{j \neq i} (Z_{ij} + Z_{ji})/(n-1), \qquad i = 1, 2, \cdots n;$$

$$\bar{T} = \sum_{i=1}^{n} T_i/n \ ; \qquad \hat{\nu} = \sum_{i=1}^{n} (T_i - \bar{T})^2/n.$$

It is known that $n^{\frac{1}{2}}\left[\frac{R}{n(n-1)} - p\right]/\hat{\nu}^{\frac{1}{2}}$ approximates to the standard normal distribution, where $p = EZ_{ij} = P(X_i > Y_j)$, $i \neq j$ [5]. When $n^{\frac{1}{2}}\left[\frac{R}{n(n-1)} - \frac{1}{2}\right]/\hat{\nu}^{\frac{1}{2}} > \Phi^{-1}(1-\alpha)$,

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we reject hypothesis H_0 with probability α, and the test is called the R-test [5]. When a bivariate continuous distribution is a bivariate normal distribution, the paired-t-test is a UMPU [1].

To see if the power of the R-test has some loss, Raviv introduced Pitman's relative efficiency and calculated it:

$$ARE(R,t) = 12\mathrm{Var}_0(x)\left\{\int f^2(x)\,dx\right\}^2 \cdot \left[\frac{1-\rho_0(X,Y)}{1-\rho_0\{F(x),F(y)\}}\right],$$

where $f(x)$ is the density function of X, ρ_0 is the correlation coefficient under H_0. So when we use the R-test to test the equivalence of marginal distributions of a BVED, we are interested in calculating Pitman's relative efficiency of the R-test against the t-test, i.e., ARE(R, t).

Change the BVED to the following:

$$\begin{aligned} F(x,y) &= \exp\{-\lambda_1 x - (\lambda_1+\theta)y - \lambda_0 \max(x,y)\} \\ &= \exp\{-\lambda_1 x - (1+\theta/\lambda_1)y - \lambda_0/\lambda_1 \cdot \max(\lambda_1 x, \lambda_1 y)\} \end{aligned} \qquad (\theta)$$

From the above, we can see that the R-test and the t-test do not depend on parameter λ_1, so we can let $\lambda_1 = 1$ (where $\lambda_2 = \lambda_1 + \theta$), and the test

$$H_0: \lambda_1 = \lambda_2 \longleftrightarrow H_1: \lambda_1 < \lambda_2$$

can be changed to the test:

$$H_0: \theta = 0 \longleftrightarrow H_1: \theta > 0.$$

Furthermore, $f(x) = \nu_1 e^{-\nu_1 x}$ (the density function of x), $x > 0$, and

$$\int_0^\infty f^2(x)\,dx = \int_0^\infty \nu_1^2 \exp(-2\lambda_1 x)\,dx = \frac{\nu_1}{2},$$

$$\rho_0(X,Y) = \frac{\lambda_0}{\lambda}, \qquad \text{where} \begin{cases} \nu_1 = \lambda_1 + \lambda_0 = 1 + \lambda_0, \\ \lambda = \lambda_1 + \lambda_1 + \lambda_0 = 2 + \lambda_0. \end{cases}$$

Assume that $F(X)$ is the distribution function of X. Then

$$F(X) = 1 - e^{-\nu_1 X}, \quad F_0(Y) = 1 - e^{-\nu_1 Y};$$

$$\begin{aligned} E_0 F(X) F(Y) &= \int_0^\infty \int_0^\infty (1-e^{-\nu_1 x})(1-e^{-\nu_1 y})\,dF(x,y) \\ &= \frac{1+\lambda_0}{4+3\lambda_0} \quad [2], \end{aligned}$$

$$E_0 F(X) = E_0 F(Y) = 1/2,$$
$$D_0 F(X) = D_0 F(Y) = 1/12.$$

Thus,

$$\rho_0\{F(X), F(Y)\} = \frac{\dfrac{1+\lambda_0}{4+3\lambda_0} - \dfrac{1}{4}}{\dfrac{1}{12}}$$

$$= \frac{12(1+\lambda_0)}{4+3\lambda_0} - 3.$$

And finally,

$$\mathrm{ARE}(R,t) = 12 \cdot \frac{1}{(1+\lambda_0)^2}\left(\frac{1+\lambda_0}{2}\right)^2 \cdot \frac{1-\dfrac{\lambda_0}{\lambda_0+2}}{1+3-\dfrac{12(1+\lambda_0)}{4+3\lambda_0}}$$

$$= \frac{9}{2}\left(1-\frac{2}{3\lambda_0+6}\right).$$

Therefore, $\min_{\lambda_0 \geq 0} \mathrm{ARE}(R,t) = 3.$

3. Simulation

We drew 50 groups of samples of size 40 from a BVED model. The table indicates the rejected proportion number by the t-test and the R-test, respectively. When L = 0, this is the test for equivalence of marginal distributions. The R-test is somewhat more conservative than the t-test, i. e., type I error probability of the t-test is larger than that of the R-test, but the difference is small.

Table

The rejected proportion number of 50 groups
of samples by the t-test and R-test
$p1 = t_{78}(0.05) = 1.668$
$p2 = \Phi^{-1}(1-0.05) = \Phi^{-1}(0.95) = 1.64$

λ_0	L*	t	R
0.1	0	0.08	0
	0.2	0.2	0.14
	0.4	0.34	0.28
	0.6	0.54	0.3
0.2	0	0.1	0.02
	0.2	0.16	0.04
	0.4	0.26	0.18
	0.6	0.56	0.38
0.8	0	0.02	0
	0.2	0.1	0
	0.4	0.1	0
	0.6	0.2	0.1

* L indicates θ in (//).

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