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<td>Author(s)</td>
<td>Inami, Tadao</td>
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Theorems on number of "trees" with a given number of "knots" or "branches" and on "spanning graphs"

By
Tadao INAMI*

1. The number of trees with \( n \) branches or with \( m \) knots.

A tree with \( n \) branches has either 1, 2, 3, \cdots, or \( n \) main branches. If the tree has one main branch, it can only be formed by adding on to this main branch a tree with \( n-1 \) branches. If the tree has two main branches, then \( p+q \) being a partition of \( n-2 \), the tree can be formed by adding onto one main branch a tree of \( p \) branches, and to the other main branch a tree of \( q \) branches, the number of trees so obtained is

\[
A_pA_q \quad \text{if} \quad p \neq q
\]

\[
\frac{1}{2}A_p(A_p+1) \quad \text{if} \quad p=q=\frac{1}{2}(n-2)
\]

where \( A_p \) and \( A_q \) are the numbers of trees that can be formed by \( p \) and \( q \) branches, respectively.

If the tree has three main branches, then if \( p+q+r \) is any partition of \( n-3 \), \( A_n \) contains the part

\[
A_pA_qA_r \quad \text{if} \quad p \neq q \neq r
\]

\[
\frac{1}{2}A_p(A_p+1)A_r \quad \text{if} \quad p=q \neq r
\]

\[
\frac{1}{6}A_p(A_p+1)(A_p+2) \quad \text{if} \quad p=q=r=\frac{1}{3}(n-3)
\]

The preordering rule for the formation of the number \( A_n \) is completely expressed by the "generating function":

\[
a(x) = A_0 + A_1x + A_2x^2 + \cdots = (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-4}\cdots
\]

Comparing the expanded right hand with the middle section of the equation, the number \( A_n \) of tropologically distinct trees with \( n \) branches is obtained. The number of topologically distinct rooted trees, \( C_n \), with \( n \) knots is equal to the number of topologically distinct trees with \( n-1 \) branches.

\[
C(x) = C_1x + C_2x^2 + C_3x^3 + \cdots = x(1-x^{-c_1})(1-x^{-c_2})(1-x^{-c_3})\cdots
\]

\[
= x(1+C_1x^2+C_1(C_1+1)x^3+\frac{C_1(C_1+1)(C_1+2)}{2!}x^4+\cdots)
\]

\[
\times (1+C_2x^2+C_2(C_2+1)x^3+\cdots)\times \cdots
\]

giving

\[
C_1 = 1
\]

\[
C_2 = C_1 + 1
\]

\[
C_3 = \frac{C_1(C_1+1)}{2!} + C_2 = 2
\]

\[
C_4 = \frac{C_1(C_1+1)(C_1+2)}{3!} + C_1C_2 + C_3 = 4
\]

\[
C_5 = \frac{C_1(C_1+1)(C_1+2)(C_1+3)}{4!} + \frac{C_1(C_1+1)(C_1+2)}{2!} + C_1C_3 + C_2 + \frac{C_1(C_1+1)}{2!} = 9
\]

* Agriculture, Home Economics and Engineering Division, University of the Ryukyus.
Thus

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>9</td>
<td>20</td>
<td>48</td>
<td>115</td>
<td>286</td>
<td>719</td>
<td>1842</td>
<td>4766</td>
<td>12486</td>
</tr>
<tr>
<td>$C_n$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>9</td>
<td>20</td>
<td>48</td>
<td>115</td>
<td>286</td>
<td>719</td>
<td>1842</td>
<td>4766</td>
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</table>

The generating function $c(x)$ may be written

$$c(x) = xe^{c(x) + c(x^2) + c(x^3) + \ldots}$$

$$= 2 \left[ 1 + \frac{c(x)}{1!} + \frac{c(x)^2 + c(x^2)}{2!} + \frac{c(x)^3 + 3c(x)c(x^2) + 2c(x^3)}{3!} + \ldots \right]$$

Fig. 1. $A_n$

Fig. 2. $C_n$

Both forms of expansion have their advantages: The first form serves as the starting point for asymptotic computation of $C_n$ and the number, $C_n'$, of topologically distinct unrooted trees with $n$ knots. The second serves as the starting point for generalization [In the general term of the series, the cycle-index of the symmetrical groups of $n$ elements is recognized.]

The number of trees $D_n$ which can be formed with $n$ given knots labelled $\alpha, \beta, \gamma, \cdots$ is given by $n^{n-2}$ (See Fig. 3)

2. Number of trees with a given number of free branches

The number of trees $B_n$ with a given number, $n$, of free branches, bifurcations at least, is given, according to Cayley

$$(1 - x)^{-1}(1 - x^{-2})^{-B_2}(1 - x^{-3})^{-B_3} \cdots = 1 + x + 2B_2x^2 + 2B_3x^3 + \cdots$$

to give, for $n = 2, 3, \ldots, 7$,

<table>
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<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_n$</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>12</td>
<td>33</td>
<td>90</td>
</tr>
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</table>
3. Multiple-operators and Labelled trees.

If U is an operand and P, Q, R, \ldots are operators, then there exists a certain relationship between decomposition of multiple-operators and trees labelled by operators and the operand, as shown in Fig. 5; where \( Q \times P \) denotes the mere algebraic product of Q and P (so does the bifurcations of branches Q and P), while QP (and the succession of Q and P nodes in cascade) denotes the result of operation performed upon P as operand.

\[
PQ = (Q \times P)U + (QP)U
\]

Fig. 5. The relationship between multiple-operators and labelled trees where no transposition of order \( PQR \ldots \) occurs from root to free branches.

4. Number of bifurcation-trees with \( n \) end-points.

By a bifurcation-tree, we mean a tree with non-terminal knots of three branches. The number \( D_n \) of bifurcation tree with a given number, \( n \), of end points is, according to Cayley, expressed in terms of \( D_1, D_2, \ldots, D_{n-1} \), as

\[
D_n = D_1D_{n-1} + D_2D_{n-2} + \cdots + D_{n-1}D_1
\]

if we let, arbitrarily

\[
D_1 = 1.
\]

If we consider a function

\[
f(x) = D_1 + xD_2 + x^2D_3 + \cdots
\]

we have

\[
f^2 = D_1D_1 + x(D_1D_3 + D_2D_2) + x^2(D_1D_3 + D_2D_3 + D_3D_1) + \cdots
\]

Thus

\[
xf^2 = f - 1
\]

\[
\therefore f = \frac{1 - \sqrt{1 - 4x}}{2x}
\]

But
\[
\sqrt{1-4x} = 1 - \frac{1}{2} + \frac{1}{2} \left( \frac{1}{2} \right) (4x)^2 - \frac{1}{2} \left( \frac{1}{2} \right) \left( \frac{-3}{2} \right) (4x)^3 + \cdots
\]
\[
= 1 - 2x - 2x^2 - 4x^3 - 10x^4 - \cdots
\]
\[
f = 1 + x + 2x^2 + 5x^3 + \cdots
\]

The coefficients of \(x^n, x^1, x^2, x^n, \cdots\) are equal to \(D_1, D_2, D_3, D_4, \cdots\). The expression for general term is seen to be

\[
D_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots (2n-3)}{1 \cdot 2 \cdot 3 \cdot \cdots n} 2^{n-1}
\]
giving

<table>
<thead>
<tr>
<th>(n)</th>
<th>(D_n)</th>
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<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
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<tr>
<td>3</td>
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<td>4</td>
<td>5</td>
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<tr>
<td>5</td>
<td>14</td>
</tr>
<tr>
<td>6</td>
<td>42</td>
</tr>
<tr>
<td>7</td>
<td>132</td>
</tr>
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</table>

Illustrations for \(n=2, 3, 4\) are shown in Figs. 6, 7, 8, respectively.

Fig. 6. \(D_2\)  
Fig. 7. \(D_3\)  
Fig. 8. \(D_4\)

If \(A, B, C, D\) are symbols capable of successive binary combinations, but do not satisfy the associative law, number of the different significations of the ambiguous expression \(ABC \cdots N\) is equal to \(D_n\). For instance, \(AB\) has only one meaning; \(ABC\) may mean either \(A \cdot BC\) or \(AB \cdot C\); \(ABCD\) may mean \(A(B \cdot CD), AB \cdot CD, (AB-C)D, (A \cdot BC)D, \) or \(A(BC \cdot D)\).

5. Spanning graphs.

Def. Spanning sub-graph. A sub-graph spans a graph if it contains all the vertices of the graph.

Def. Maximal graph. A graph is maximal if it is not contained in any larger graph of the same sort.

Def. Minimal graph. A graph is minimal if it does not contain any smaller graph of the same sort.

Def. Forest. A forest is a graph with no loops.

Problem. Give a practical method for constructing a spanning subtree of minimum length.

Solution. There is no loss of generality in assuming that the given connected graph \(G\) is complete, i.e., that every pair of vertices is connected by an edge. For if any edge of \(G\) is "missing", it is possible to consider the "missing" edge as an edge of infinite length.

Construction A. Perform the following step as many times as possible: Among the edges of \(G\) not yet chosen, choose the shortest edge which does not form any loops with those edges already chosen. Clearly the set of edges eventually chosen must form a spanning tree of \(G\), and in fact it forms a shortest spanning tree.

Construction A'. (Dual of A) Perform the following step as many times as possible: Among the edges not yet chosen, choose the longest edge whose removal will not disconnect them. Clearly the set of edges not eventually chosen forms a spanning tree of \(G\), and in fact it forms a shortest spanning tree.

Construction B. Let \(V\) be an arbitrary but fixed (nonempty) subset of the vertices of \(G\). Then perform the following step as many times as possible: Among the edges of \(G\) which are not yet chosen but which are connected either to a vertex of \(V\) or to an edge
already chosen, pick the shortest edge which does not form any loops with the edges already chosen. Clearly the set of edges eventually chosen forms a spanning tree of $G$, and in fact it forms a shortest spanning tree. In case $V$ is the set of all vertices of $G$, then Construction $B$ reduces to Construction $A$.

Theorem 1. If $G$ is a connected graph with $n$ vertices, and $T$ is a subgraph of $G$, then the following conditions are all equivalent:

(a) $T$ is a spanning tree of $G$;
(b) $T$ is a maximal forest in $G$;
(c) $T$ is a minimal connected spanning graph of $G$;
(d) $T$ is a forest with $n-1$ edges.
(e) $T$ is a connected spanning graph with $n-1$ edges.

Theorem 2. If the edges of $G$ all have distinct lengths, then $T$ is unique, where $T$ is any shortest spanning tree of $G$.

Proof. In Construction $A$ in the above problem, let the edges chosen be called $a_1, \ldots, a_{n-1}$ in the order chosen. Let $A_i$ be the forest consisting of edges $a_1$ through $a_i$. From the hypothesis that the edges of $G$ have distinct lengths, it is easily seen that Construction $A$ proceeds in a unique manner. Thus the $A_i$ are unique, and hence also $T$.

Suppose $T \neq A$. Let $a_i$ be the first edge of $A_{n-1}$ which is not in $T$. Then $a_i, \ldots, a_{i-1}$ are in $T$. $TU a_i$ must have exactly one loop, which must contain $a_i$. This loop must also contain some edge $e$ which is not in $A_{n-1}$. Then $TU a_i - e$ is a forest with $n-1$ edges.

As $A_{i-1} U e$ is contained in the last named forest, it is a forest, so from Construction $A$, $\text{length}(e) > \text{length}(a_i)$

But then $TU a_i - e$ is shorter than $T$. This contradicts the definition of $T$, and hence proves indirectly that $T = A_{n-1}$. Q. E. D.

References

与られた数の「節」や「枝」をもった「木」の数および「包括図」に関する諸定理 (摘要)

伊波直朗

この論文では、n 個の「枝」または m 個の「節」をもったトポロジー的に異なる「木」の数、α, β, γ,
…とレッテルをはってある n 個の「節」をもったトポロジー的に異なる「木」の数、m 個の「枝
端」をもったトポロジー的に異なる「木」の数、α 個の「枝端」をもったトポロジー的に異なる「分岐木」の数、等に関する諸定理を提起、証明した。

n 個の「枝」をもったトポロジー的に異なる「木」の数 A_n および n 個の「節」をもったトポロ
ジー的に異なる「木」の数 C_n は、それぞれ次の「生成関数」a(x), c(x) によってあらわすことができると。

\[ a(x) = A_0 + A_1 x + A_2 x^2 + \cdots = (1 - x) - (1 - x^2) - A_1 (1 - x^3) - A_2 \cdots \]

\[ c(x) = C_0 + C_1 x + C_2 x^2 + \cdots = x (1 - x) - C_1 (1 - x^2) - C_2 \cdots \]

すなわち、n=1, 2, 3, …, 12 に対する A_n の値は、1, 2, 4, 9, 20, 48, 115, 286, 719, 1842, 4766、
12486 で、C_n の値は、1, 1, 2, 4, 9, 10, 48, 115, 286, 719, 1278, 4766 である。

n 個の「枝端」をもったトポロジー的に異なる「木」の数 B_n は次の「生成関数」b(x) によってあ
らわすことができる。

\[ b(x) = (1 - x) - (1 - x^2) - B_1 (1 - x^3) - B_2 \cdots = 1 + x + 2B_2 x^2 + 2B_3 x^3 + \cdots \]

すなわち、n=1, 2, 3, …, 9 に対する B_n の値はそれぞれ 0, 1, 2, 5, 9, 33, 90 である。

n 個の「枝端」をもったトポロジー的に異なる「分岐木」の数 D_n は次の「生成関数」d(x) によって
あらわすことができる。

\[ d(x) = D_0 + D_1 x + D_2 x^2 + \cdots = \frac{1 - \sqrt{1 - 4x}}{2x} \]

すなわち、n=1, 2, 3, …, 7 に対する D_n の値は、1, 1, 2, 5, 14, 42, 132 である。

「図」の全頂点を含む「節分図」はその図を「包括する」という。同一種類のそれより大きな「図」
に含まれない「図」は「最大」であるという。同一種類のそれより小さな「図」を含まない「図」は
「最小」であるという。ループを含まない「図」を「森」という。

上の定義にしたがえば、次の定理が成立する。

定理 1 もし G が n 個の頂点をもつ連結した「図」であり、T が G の部分図であれば、次の条件
は等価である。

(a) T は G の包括木である。
(b) T は G の最大木である。
(c) T は G の最小連結包括図である。
(d) T は n−1 個の枝をもった木である。
(e) T は n−1 個の枝をもった連結包括図である。

定理 2 G の枝が全部つながった長さであれば前定理の条件を満足する T は一意的にさだまう。こ
のとき T は G の任意の最短包括木である。

G の最短包括木を作るに当っての実際的な方法も示してある。