



Title	HIRATA SEPARABLE EXTENSION OVER THE DESCENDING CHAIN OF SUBRINGS
Author(s)	Yamashiro, Yasukazu
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HIRATA SEPARABLE EXTENSION OVER THE DESCENDING CHAIN OF SUBRINGS

Y. YAMASHIRO

Abstract

In [5], we considered the equivalent class of descending chains of subrings.

In this note, we consider Hirata separable extension (H-separable extension) over the descending chain of subrings. Let Λ be a ring, Γ a subring of Λ , C the center of Λ and Δ the commutator of Γ in Λ . Λ is a Hirata separable extension over Γ if and only if Δ is C -finitely generated projective and $\Lambda \otimes_{\Gamma} \Lambda \cong \text{Hom}_C(\Delta, \Lambda)$ by the map $a \otimes b \mapsto [d \mapsto adb]$, where $a \otimes b \in \Lambda \otimes_{\Gamma} \Lambda$ and $d \in \Delta$ (Theorem 1.1 in [3]).

1. TENSOR PRODUCT OVER THE DESCENDING CHAIN OF SUBRINGS

Throughout this note, Λ is a ring, Ω is a class of subrings of Λ with Λ . We can regard Ω as a poset under the inclusion relation. Now we consider the set of all monotone functions of \mathbb{N} to Ω^{op} , denoted by $\mathcal{C}(\Omega)$, where \mathbb{N} is the poset of all natural numbers. Throughout this note, B and B' are members of $\mathcal{C}(\Omega)$. $\mathcal{C}(\Omega)$ is also a poset by taking the relation $B \leq B'$, if $Bi \leq B'i$ in Ω , for every $i \in \mathbb{N}$. We will write $B \preceq B'$ if there exists a monotone function ρ of \mathbb{N} to \mathbb{N} such that $B\rho \leq B'$. Then $\mathcal{C}(\Omega)$ is a preordered set and if we define the relation $B \sim B'$ when $B \preceq B'$ and $B' \preceq B$, the relation " \sim " is an equivalence relation and the set of equivalent classes of the relation " \sim " is a poset. Use $\tilde{\mathcal{C}}(\Omega)$ denote this poset. We can observe that Ω is contained in $\tilde{\mathcal{C}}(\Omega)$ (cf. [5]).

For any $i \in \mathbb{N}$, we will call left (right) Bi -module simply left (right) B -module. If $B \preceq B'$ then B' -module is B -module. So we can define $[B]$ -module, for any

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$[B] \in \mathcal{C}(\Omega)$. Let N and M be left B -module and right B -module, respectively, then there exist numbers i and j such that N is left B^i -module and M is right B^j -module. Put $\max\{i, j\} = k$. Then for any $l \geq k$, M and N are B^l -module, as left and as right, respectively. So we have the sequence:

$$M \otimes_{B^k} N \leftarrow M \otimes_{B^{(k+1)}} N \leftarrow \cdots$$

where homomorphisms are canonical epimorphisms: $m \otimes n \mapsto m \otimes n$. Now we consider the inverse limit $\varprojlim_l M \otimes_{B^l} N$, denoted by $M \otimes_{\bar{B}} N$. We will denote $\bigcap_l B^l = \bar{B}$. Then there exist the canonical epimorphisms $M \otimes_{\bar{B}} N \rightarrow M \otimes_{B^l} N : m \otimes n \mapsto m \otimes n$, and the epimorphism $M \otimes_{\bar{B}} N \rightarrow M \otimes_B N$. So we may write the element of $M \otimes_B N$, $\sum_s m_s \otimes n_s$, for some elements m_s of M and some elements n_s of N . Let $B \preceq B'$ and ρ the monotone function of \mathbb{N} to \mathbb{N} for $B \preceq B'$. Then for any $l \in \mathbb{N}$, there exist the composition homomorphism $M \otimes_B N \rightarrow M \otimes_{B^{\rho l}} N \rightarrow M \otimes_{B'^l} N : m \otimes n \mapsto m \otimes n$. So there exists the homomorphism $M \otimes_B N \rightarrow M \otimes_{B'} N : m \otimes n \mapsto m \otimes n$. Therefore we obtain the following statement.

Proposition 1. *If $B \sim B'$ then $M \otimes_B N$ is isomorphic to $M \otimes_{B'} N$.*

2. HIRATA SEPARABLE EXTENSION

For a two-sided Λ -module M and a subset S of Λ , we denote the commutator of S in M by $V_M(S)$:

$$V_M(S) = \{m \in M \mid sm = ms \ (\forall s \in S)\}.$$

If Γ is a subring of Λ , then $\text{Hom}({}_\Lambda \Lambda \otimes_\Gamma \Lambda_\Lambda, {}_\Lambda \Lambda_\Lambda)$ is isomorphic to $V_\Lambda(\Gamma)$ by the map $f \mapsto f(1 \otimes 1)$. So we have the isomorphism

$$\begin{aligned} \text{Hom}({}_\Lambda \Lambda \otimes_B \Lambda_\Lambda, {}_\Lambda \Lambda_\Lambda) &= \text{Hom}({}_\Lambda \varprojlim_l \Lambda \otimes_{B^l} \Lambda_\Lambda, {}_\Lambda \Lambda_\Lambda) \\ &\cong \varprojlim_l \text{Hom}({}_\Lambda \Lambda \otimes_{B^l} \Lambda_\Lambda, {}_\Lambda \Lambda_\Lambda) \cong \varprojlim_l V_\Lambda(B^l) \cong V_\Lambda(\bar{B}). \end{aligned}$$

We will put that $\Delta = V_\Lambda(\bar{B})$ and C is the center of Λ , namely $C = V_\Lambda(\Lambda)$

Definition 1. Λ is Hirata separable extension over B if Δ is C -finitely generated projective and $\Lambda \otimes_B \Lambda \cong \text{Hom}_C(\Delta, \Lambda)$ by the map $a \otimes b \mapsto [d \mapsto adb]$, where $a \otimes b \in \Lambda \otimes_B \Lambda$ and $d \in \Delta$.

If $B \sim B'$ then $\bar{B} = \bar{B}'$. And so Λ is Hirata separable extension over B if and only if Λ is Hirata separable extension over B' .

Now we can state the similar proposition of Proposition 1 in [4].

Proposition 2. Λ is Hirata separable extension over B if and only if there exist $\delta_i \in \Delta$ ($i = 1, 2, \dots, n$) and $\sum_j a_{ij} \otimes b_{ij} \in V_{\Lambda \otimes_B \Lambda}(\Lambda)$ such that

$$1 \otimes 1 = \sum_{ij} \delta_i a_{ij} \otimes b_{ij}$$

in $\Lambda \otimes_B \Lambda$.

Proof. $\text{Hom}({}_\Lambda \Lambda \otimes_B \Lambda_\Lambda, {}_\Lambda \Lambda_\Lambda) \cong \Delta$ and $\text{Hom}({}_\Lambda \Lambda_\Lambda, {}_\Lambda \Lambda \otimes_B \Lambda_\Lambda) \cong V_{\Lambda \otimes_B \Lambda}(\Lambda)$ by the map $f \mapsto f(1 \otimes 1)$ and $g \mapsto g(1)$, respectively.

Suppose Λ is Hirata separable extension over B . Since

$$\text{Hom}_C(\Delta, \Lambda) \subset \oplus \text{Hom}_C(\oplus_n C, \Lambda) \cong \oplus_n \Lambda$$

as two-sided Λ -module, the assertion holds.

Conversely, suppose that there exist $\delta_i \in \Delta$ and $\sum_j a_{ij} \otimes b_{ij} \in V_{\Lambda \otimes_B \Lambda}(\Lambda)$ such that $1 \otimes 1 = \sum_{ij} \delta_i a_{ij} \otimes b_{ij}$ in $\Lambda \otimes_B \Lambda$. Then $\Lambda \otimes_B \Lambda \subset \oplus (\oplus_n \Lambda)$ as two-sided Λ -module and

$$\begin{aligned} \Delta &\cong \text{Hom}({}_\Lambda \Lambda \otimes_B \Lambda_\Lambda, {}_\Lambda \Lambda_\Lambda) \subset \oplus \text{Hom}({}_\Lambda (\oplus_n \Lambda)_\Lambda, {}_\Lambda \Lambda_\Lambda) \\ &\cong \oplus_n \text{Hom}({}_\Lambda \Lambda_\Lambda, {}_\Lambda \Lambda_\Lambda) \cong \oplus_n C. \end{aligned}$$

Therefore Δ is C -finitely generated projective.

Moreover there exists homomorphism $\text{Hom}_C(\Delta, \Lambda) \rightarrow \Lambda \otimes_B \Lambda$ by the map $f \mapsto \sum_{ij} f(\delta_i) a_{ij} \otimes b_{ij}$. Then the composition map $\Lambda \otimes_B \Lambda \rightarrow \text{Hom}_C(\Delta, \Lambda) \rightarrow \Lambda \otimes_B \Lambda$ and $\text{Hom}_C(\Delta, \Lambda) \rightarrow \Lambda \otimes_B \Lambda \rightarrow \text{Hom}_C(\Delta, \Lambda)$ are identity maps. \square

We denote the set of δ_i in above proposition by δ .

Corollary 1. *If Λ is Hirata separable extension over B and $\delta \subset V_\Lambda(Bk)$ for some k . Then Λ is Hirata separable extension over B_l , for any $l \geq k$.*

Proof. By the epimorphism $\Lambda \otimes_B \Lambda \rightarrow \Lambda \otimes_{B_l} \Lambda$, $V_{\Lambda \otimes_B \Lambda}(\Lambda)$ is mapped into $V_{\Lambda \otimes_{B_l} \Lambda}(\Lambda)$. So $\sum_j a_{ij} \otimes b_{ij}$ in above proposition are belong to $V_{\Lambda \otimes_{B_l} \Lambda}(\Lambda)$. By the application Proposition 1 in [4] to Λ and B_l , Λ is a Hirata separable extension over B_l . \square

Corollary 2. *Let $\bar{B} = V_\Lambda(\Delta)$. If Λ is Hirata separable extension over B and $Bk \subset \bar{B}$, for some k . Then Λ is a Hirata separable extension over \bar{B} .*

Proof. The composition homomorphism $\Lambda \otimes_B \Lambda \rightarrow \Lambda \otimes_{Bk} \Lambda \rightarrow \Lambda \otimes_{\bar{B}} \Lambda$ maps $V_{\Lambda \otimes_B \Lambda}(\Lambda)$ into $V_{\Lambda \otimes_{\bar{B}} \Lambda}(\Lambda)$. The rest of proof is same to that of Corollary 1. \square

Λ is a strongly separable extension over a subring Γ if and only if there exist $\delta_i \in V_\Lambda(\Gamma)$ ($i = 1, 2, \dots, n$) and $\sum_j a_{ij} \otimes b_{ij} \in V_{\Lambda \otimes_\Gamma \Lambda}(\Lambda)$ such that $d = \sum_{ij} \delta_i a_{ij} db_{ij}$ for any $d \in V_\Lambda(\Gamma)$ (Theorem 3.5 in [2]).

Theorem 1. *If Λ is Hirata separable extension over B and the epimorphism $\Lambda \otimes_{\bar{B}} \Lambda \rightarrow \Lambda \otimes_B \Lambda$ maps $V_{\Lambda \otimes_{\bar{B}} \Lambda}(\Lambda)$ onto $V_{\Lambda \otimes_B \Lambda}(\Lambda)$. Then Λ is a strongly separable extension over \bar{B} .*

Proof. Let $\delta_i \in \Delta$ and $\sum_j a_{ij} \otimes b_{ij} \in V_{\Lambda \otimes_B \Lambda}(\Lambda)$ in Proposition 2. Then $\delta_i \in V_\Lambda(\bar{B})$ and, by assumption, in $\Lambda \otimes_{\bar{B}} \Lambda$, $\sum_j a_{ij} \otimes b_{ij}$ may be belong to $V_{\Lambda \otimes_{\bar{B}} \Lambda}(\Lambda)$. So there exists the homomorphism $\text{Hom}_C(\Delta, \Lambda) \rightarrow \Lambda \otimes_{\bar{B}} \Lambda$ by $f \mapsto \sum_{ij} f(\delta_i) a_{ij} \otimes b_{ij}$.

On the other hand, the homomorphism $\Lambda \otimes_{\bar{B}} \Lambda \rightarrow \text{Hom}_C(\Delta, \Lambda)$ is equal to the composition of $\Lambda \otimes_{\bar{B}} \Lambda \rightarrow \Lambda \otimes_B \Lambda \rightarrow \text{Hom}_C(\Delta, \Lambda)$. We can observe that the composition of $\text{Hom}_C(\Delta, \Lambda) \rightarrow \Lambda \otimes_{\bar{B}} \Lambda \rightarrow \text{Hom}_C(\Delta, \Lambda)$ maps the identity map to $[d \mapsto \sum_{ij} \delta_i a_{ij} db_{ij}]$, and the composition of $\text{Hom}_C(\Delta, \Lambda) \rightarrow \Lambda \otimes_{\bar{B}} \Lambda \rightarrow \Lambda \otimes_B \Lambda \rightarrow \text{Hom}_C(\Delta, \Lambda)$ maps the identity map to $[d \mapsto d]$. Therefore $d = \sum_{ij} \delta_i a_{ij} db_{ij}$ for any $d \in \Delta$ and Λ is a strongly separable extension of \bar{B} . \square

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Faculty of Education
University of the Ryukyus
Nishihara-cho, Okinawa 903-0213
JAPAN