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<td>Author(s)</td>
<td>Nakazato, Haruo</td>
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<td>Citation</td>
<td>琉球大学理工学部紀要 (理学編) = Bulletin of Science &amp; Engineering Division, University of Ryukyus. Mathematics &amp; natural sciences (24): 21-27</td>
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<td>Issue Date</td>
<td>1977-09</td>
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<td>URL</td>
<td><a href="http://hdl.handle.net/20.500.12000/24498">http://hdl.handle.net/20.500.12000/24498</a></td>
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On \( \mathfrak{F} \)-reducers in Finite Solvable Groups

Haruo NAKAZATO*

1. INTRODUCTION. In this note all groups are finite and solvable. The letter \( G \) stands always for such a group. A. Mann constructed in [3], for any subgroup \( H \) of \( G \), a subgroup \( Q(H) \) which gives a different characterization of the reducer \( R(H) \) in \( G \) of \( H \), defined by B. Fischer, and he defined in [4] another subgroup \( M(H) \) by using a certain concept of equivalence introduced by R. Carter. In [5] he provided an alternative characterization of the Carter subgroups of \( G \) as nilpotent subgroups \( H \) of \( G \) satisfying \( H= M(H) \).

In [1] C. J. Graddon introduced the concept of the \( \mathfrak{F} \)-reducer \( R(H; \mathfrak{F}) \) in \( G \) of a subgroup \( H \) of \( G \) by defining \( \mathfrak{F} \)-basis of \( G \), which gives an alternative characterization of \( Q(H; \mathfrak{F}) \) which is a generalization of the work of A. Mann [3], and showed some of the basic properties of this subgroup, where \( \mathfrak{F} \) is the local (saturated) formation defined by a set of nonempty subgroup closed formations \( \{ \mathfrak{F}(p) \} \). He showed in [1] that the \( \mathfrak{F} \)-projector of \( G \) are characterized as the \( \mathfrak{F} \)-subgroup \( H \) of \( G \) satisfying \( H= R(H; \mathfrak{F}) \).

In this note we give, for a certain subgroup \( H \) of \( G \), an alternative characterization of the \( \mathfrak{F} \)-reducer \( R(H; \mathfrak{F}) \) of \( H \) in \( G \) as the subgroup \( M(H; \mathfrak{F}) \) which is similar to the subgroup \( M(H) \) introduced by A. Mann, and show some properties of \( \mathfrak{F} \)-reducer \( R(H; \mathfrak{F}) \) of \( H \) in \( G \). In section 2 we give a brief resume of the definitions and properties which we require later in this note, and in section 3 we show some properties of \( \mathfrak{F} \)-subnormal subgroups of \( G \).

2. PRELIMINARIES. We shall wherever possible, adhere to the notation used in [1]. Throught this note, \( \mathfrak{F} \) will denote the integrated formation defined locally by the nonempty subgroup closed formations\( \{ \mathfrak{F}(p) \} \). Let \( \{ S^p \} \) be a set of Sylow \( p \)-complements of \( G \), one for each prime \( p \) dividing \( |G| \), and let \( \mathfrak{S} \) be a Sylow system of \( G \) generated by the \( S^p \). Then the \( \mathfrak{F} \)-basis of \( G \) associated with \( \mathfrak{S} \) is the collection \( \mathfrak{F}(\mathfrak{S})= \{ S^p \cap G_\mathfrak{F}(p) \} \) of subgroups of \( G \), where for each prime \( p \), \( G_\mathfrak{F}(p) \) denotes the \( \mathfrak{F}(p) \)-residual of \( G \), i.e., the smallest normal subgroup of \( G \) with the factor in \( \mathfrak{F}(p) \). Let \( H \) be a subgroup of \( G \), then, as in [1], \( \mathfrak{F}(\mathfrak{S}) \) reduces into \( H \) if for each prime \( p \), \( S^p \cap H_\mathfrak{F}(p)= S^p \cap G_\mathfrak{F}(p) \cap H_\mathfrak{F}(p) \) is a Sylow \( p \)-complement of \( H_\mathfrak{F}(p) \), i.e., if \( S^p \cap H_\mathfrak{F}(p) \) is an \( \mathfrak{F} \)-basis of \( H \).

Thus \( \mathfrak{F}(\mathfrak{S}) \) reduces into \( H \) if and only if there exists a Sylow system \( \mathfrak{S}_H= \{ H^p \} \) of \( H \) such that \( S^p \cap H_\mathfrak{F}(p)= H^p \cap H_\mathfrak{F}(p) \) for each prime \( p \). In [1] C. J. Graddon showed

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that there always exists at least one $\mathfrak{F}$-basis of $G$ which reduces into $H$ and defined, for given such an $\mathfrak{F}$-basis $\mathfrak{F}(\mathcal{S})$, the $\mathfrak{F}$-reducer of $H$ in $G$ to be the subgroup

$$R(H; \mathfrak{F}) = \langle g \in G : \mathfrak{F}(\mathcal{S})^g \text{ reduces into } H \rangle$$

DEFINITION. A maximal subgroup $M$ of $G$, of index powers of a prime $p$ in $G$, is said to be $\mathfrak{F}$-normal in $G$ if $M/\text{Core}(M) \in \mathfrak{F}(p)$. $M$ is said to be $\mathfrak{F}$-abnormal otherwise. A subgroup $H$ of $G$ is $\mathfrak{F}$-abnormal in $G$ if every link in each maximal chain joining $H$ to $G$ is $\mathfrak{F}$-abnormal. $H$ is said to be $\mathfrak{F}$-subnormal in $G$ if every link in some maximal chain joining $H$ to $G$ is $\mathfrak{F}$-normal.

In [2] it is described that, for a subgroup $H$ of $G$

(2.1) $H$ is $\mathfrak{F}$-subnormal in $G$ if and only if every $\mathfrak{F}$-basis of $G$ reduces into $H$.

The following two results are showed by C. J. Graddon in [1]:

(2.2) $H$ is an $\mathfrak{F}$-abnormal subgroup of $G$ if and only if $H = R(H; \mathfrak{F})$.

(2.3) If $H$ is a subgroup of $G$, then $R(H; \mathfrak{F})$ is self $\mathfrak{F}$-reducing in $G$.

Let $\mathcal{Q}$ be the collection of $\mathfrak{F}$-bases of $G$ and let $\mathcal{M}$ be the set of elements of $\mathcal{Q}$ which reduces into the subgroup $H$ of $G$. Let $\mathcal{M}_0$ be the block generated by $\mathcal{M}$ in $\mathcal{Q}$. Then $\mathcal{Q}(H; \mathfrak{F})$ is defined to be the set stabilizer in $G$ of $\mathcal{M}_0$, i.e., the set of all elements $g$ in $G$ such that $(\mathcal{M}_0)^g = \mathcal{M}_0$.

C. J. Graddon showed in [2] that

(2.4) Every $\mathfrak{F}$-bases of $G$ which reduces into the subgroup $H$ of $G$ also reduces into $R(H; \mathfrak{F})$.

(2.5) $\mathcal{M}_0$ is the set of $\mathfrak{F}$-bases of $G$ which reduces into $R(H; \mathfrak{F})$.

and in [1] that

(2.6) For each subgroup $H$ of $G$, $R(H; \mathfrak{F}) = \mathcal{Q}(H; \mathfrak{F})$.

Let $H$ be a subgroup of $G$. Then an $H$-composition series of $G$ is a series

$$1 = G_n < G_{n-1} < \cdots < G_1 < G_0 = G$$

in which each subgroup $G_i$ is a maximal $H$-invariant normal subgroup of $G_{i-1}$. We say that the factor $G_i/G_{i+1}$ is $\mathfrak{F}$-central if $A_{\mathfrak{F}}(G_i/G_{i+1})$, the automorphism group induced by $H$ on $G_i/G_{i+1}$, belongs to the formation $\mathfrak{F}(p)$, where $G_i/G_{i+1}$ is an elementary abelian $p$-group. If this is not the case we say this factor is $\mathfrak{F}$-eccentric.

The following result is the structure theorem of $R(H; \mathfrak{F})$, which is obtained by C. J. Graddon in [2].

(2.7) Let $H$ be a subgroup of $G$. Then (i) $R(H; \mathfrak{F})$ covers each $\mathfrak{F}$-central $H$-composition factor of $G$, and (ii) if $K$ is a subgroup of $G$ which contains $H$ and covers every $\mathfrak{F}$-central $H$-composition factor of $G$, then $K$ contains $R(H; \mathfrak{F})$.

DEFINITION. Suppose that $H \leq K \leq G$. Then $K$ is an $\mathfrak{F}$-subnormalizer of $H$ in $G$ if

(i) $H$ is $\mathfrak{F}$-subnormal in $K$, and

(ii) Whenever $H$ is $\mathfrak{F}$-subnormal in a subgroup $L$ of $G$, then $L$ is contained
The following facts follows from theorem 4.6 of [2].

\[(2.8)\] If \(H\) is a subgroup of \(G\) and the set of \(\mathfrak{F}\)-bases of \(G\) which reduce into \(H\) forms a block then \(R(H;\mathfrak{F})\) is an \(\mathfrak{F}\)-subnormalizer of \(H\) in \(G\).

3. \(\mathfrak{F}\)-SUBNORMAL. We show some properties of \(\mathfrak{F}\)-subnormal subgroups of \(G\).

**Proposition 1.** Suppose that \(H \leq K \leq G\). If \(H\) is \(\mathfrak{F}\)-subnormal in \(G\), then \(H\) is \(\mathfrak{F}\)-subnormal in \(K\).

**Proof.** Let \(\mathfrak{F}(\mathcal{S}_K)\) be an \(\mathfrak{F}\)-basis of \(K\) associated with a Sylow system \(\mathcal{S}_K = \{\mathcal{K}_p\}\) of \(K\). Then there exists a Sylow system \(\mathcal{S} = \{\mathcal{S}_p\}\) of \(G\) which is an extension of \(\mathcal{S}_K\), i.e., \(\mathcal{K}_p = \mathcal{S}_p \cap K\) for each prime \(p\). Now \(\mathfrak{F}(\mathcal{S})\) is an \(\mathfrak{F}\)-basis of \(G\). Since \(H\) is \(\mathfrak{F}\)-subnormal in \(G\), \(\mathfrak{F}(\mathcal{S})\) reduces into \(H\) by (2.1). Then there exists a Sylow system \(\mathcal{S}_H = \{\mathcal{H}_p\}\) of \(H\) such that \(\mathcal{S}_p \cap \mathcal{H}_p = \mathcal{H}_p \cap \mathcal{H}_{\mathfrak{S}(p)}\), for each prime \(p\). Therefore we have that

\[
\mathcal{K}_p \cap \mathcal{H}_{\mathfrak{S}(p)} = (\mathcal{S}_p \cap K) \cap \mathcal{H}_{\mathfrak{S}(p)} = \mathcal{S}_p \cap \mathcal{H}_{\mathfrak{S}(p)} = \mathcal{H}_p \cap \mathcal{H}_{\mathfrak{S}(p)}
\]

for each prime \(p\), and thus \(\mathfrak{F}(\mathcal{S}_K)\) reduces into \(H\). This implies that \(H\) is \(\mathfrak{F}\)-subnormal in \(K\).

**Proposition 2.** Let \(H\) be a subgroup of \(G\) and suppose that \(\mathfrak{F}(\mathcal{S})\) is an \(\mathfrak{F}\)-basis of \(G\) which reduce into \(H\). If \(K\) is an \(\mathfrak{F}\)-subnormal subgroup of \(H\), then \(\mathfrak{F}(\mathcal{S})\) reduces into \(K\).

**Proof.** Since \(\mathfrak{F}(\mathcal{S}) = \{\mathcal{S}_p \cap G_{\mathfrak{S}(p)}\}\) reduce into \(H\), there exists a Sylow system \(\mathcal{S}_H = \{\mathcal{H}_p\}\) of \(H\) such that \(\mathcal{S}_p \cap \mathcal{H}_p = \mathcal{H}_p \cap \mathcal{H}_{\mathfrak{S}(p)}\), for each prime \(p\). Therefore if \(K\) is \(\mathfrak{F}\)-subnormal in \(H\), \(\mathfrak{F}(\mathcal{S}_H)\) reduces into \(K\) by (2.1), i.e., there exists a Sylow system \(\mathcal{S}_K = \{\mathcal{K}_p\}\) of \(K\) such that \(\mathcal{H}_p \cap \mathcal{K}_p = \mathcal{K}_p \cap \mathcal{K}_{\mathfrak{S}(p)}\), for each prime \(p\). Since \(\mathfrak{F}(\mathcal{P})\) is subgroup closed, we know that \(K \leq H\) implies \(K_{\mathfrak{S}(p)} \leq H_{\mathfrak{S}(p)}\). Now we have that, for each prime \(p\),

\[
\mathcal{S}_p \cap \mathcal{K}_{\mathfrak{S}(p)} = (\mathcal{S}_p \cap \mathcal{H}_{\mathfrak{S}(p)}) \cap K_{\mathfrak{S}(p)} = (\mathcal{H}_p \cap \mathcal{H}_{\mathfrak{S}(p)}) \cap K_{\mathfrak{S}(p)}
\]

Thus \(\mathfrak{F}(\mathcal{S})\) reduces into \(K\).

**Proposition 3.** Let \(H\) be a subgroup and \(N\) a normal subgroup of \(G\). Suppose that \(H\) is \(\mathfrak{F}\)-subnormal in \(G\). Then \(HN/N\) is \(\mathfrak{F}\)-subnormal in \(G/N\) and \(HN\) is \(\mathfrak{F}\)-subnormal in \(G\).

**Proof.** Now each \(\mathfrak{F}\)-basis of \(G/N\) is \(\mathfrak{F}(\mathcal{S}/N)\) for some Sylow system \(\mathcal{S}\) of \(G\), where \(\mathcal{S}/N = \{\mathcal{S}_p \cap N\}/N\) is a Sylow system of \(G/N\) for the Sylow system \(\mathcal{S} = \{\mathcal{S}_p\}\) of \(G\). Suppose that \(H\) is \(\mathfrak{F}\)-subnormal in \(G\). Then every \(\mathfrak{F}\)-bases \(\mathfrak{F}(\mathcal{S})\) of \(G\) reduce into \(H\). Therefore we have that \(\mathcal{S}_p \cap H_{\mathfrak{S}(p)} = \mathcal{H}_p \cap H_{\mathfrak{S}(p)}\) for each Sylow \(p\)-complement \(H_p\) of \(H\), and so by (2.5) and (2.6) of [1].
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\[(S^p N / N) \cap (H N / N)_{\mathfrak{F}(p)} = (S^p N / N) \cap (H_{\mathfrak{F}(p)} N / N) = (S^p N \cap H_{\mathfrak{F}(p)} N) / N = (S^p N \cap H_{\mathfrak{F}(p)} N) / N = (H^p N / N) \cap (H_{\mathfrak{F}(p)} N / N) = (H N / N)^p \cap (H N / N)_{\mathfrak{F}(p)}.\]

Therefore every $\mathfrak{F}$—basis $\mathfrak{F}(\mathcal{C} N / N)$ of $G / N$ reduces into $H N / N$. Hence $H N / N$ is $\mathfrak{F}$—subnormal in $G / N$ by (2.1).

Let $H N / N = G_r / N < \cdots < G_0 / N = G / N$

be a maximal chain joining $H N / N$ to $G / N$ such that $G / N$ is an $\mathfrak{F}$—normal maximal subgroup of $G_r / N$. Now $G_t$ is a maximal subgroup of $G_r / N$ if and only if $G_t / N$ is a maximal subgroup of $G_r / N$. On the other hand, since $\text{Core}_{G_r / N}(G / N) = \text{Core}_{G_r / N}(G / N)$, it follows that $G_t$ is $\mathfrak{F}$—normal in $G_r / N$ if and only if $G_t / N$ is $\mathfrak{F}$—normal in $G_r / N$. Therefore we have a maximal chain joining $H N$ to $G$ such that every normal link is $\mathfrak{F}$—normal:

\[H N = G_r < \cdots < G_0 = G.\]

Thus $H N$ is $\mathfrak{F}$—subnormal in $G$.

4. $\mathfrak{F}$—REDUCER. Suppose that $\mathfrak{F}$ is an integrated formation defined locally by the nonempty subgroup closed formations $\{\mathfrak{F}(p)\}$.

**Lemma 4.** Let $H \leq K \leq G$ and $\mathfrak{F}_K = \{K^p\}$ be a Sylow system of $K$. Suppose that $\mathfrak{F} = \{S^p\}$ is a Sylow system of $G$ which is an extension of $\mathfrak{F}_K$, i.e., $S^p \cap K = K^p$ for each prime $p$. Then the $\mathfrak{F}$—basis $\mathfrak{F}(\mathfrak{F}_K)$ of $K$ reduces into $H$ if and only if the $\mathfrak{F}$—basis $\mathfrak{F}(\mathfrak{F})$ of $G$ reduces into $H$.

**Proof.** Suppose that $\mathfrak{F}(\mathfrak{F}_K) = \{K^p \cap S_{\mathfrak{F}(p)}\}$ reduces into $H$. Then there exists a Sylow system $\mathfrak{F}_H = \{H^p\}$ of $H$ such that $K^p \cap H_{\mathfrak{F}(p)} = H^p \cap H_{\mathfrak{F}(p)}$ for each prime $p$. Thus we have that $S^p \cap H_{\mathfrak{F}(p)} = S^p \cap S_{\mathfrak{F}(p)} = K^p \cap H_{\mathfrak{F}(p)} = H^p \cap H_{\mathfrak{F}(p)}$ for each prime $p$. Therefore $\mathfrak{F}(\mathfrak{F})$ reduces into $H$.

Conversely, suppose that $\mathfrak{F}(\mathfrak{F}) = \{S^p \cap G_{\mathfrak{F}(p)}\}$ reduces into $H$. Then there exists a Sylow system $\mathfrak{F}_H = \{H^p\}$ of $H$ such that $S^p \cap H H^p \cap H_{\mathfrak{F}(p)} = S^p \cap H_{\mathfrak{F}(p)} = H^p \cap H_{\mathfrak{F}(p)}$ for each prime $p$. Thus we have that $K^p \cap H_{\mathfrak{F}(p)} = S^p \cap K \cap H_{\mathfrak{F}(p)} = S^p \cap H_{\mathfrak{F}(p)} = H^p \cap H_{\mathfrak{F}(p)}$ for each prime $p$. Therefore $\mathfrak{F}(\mathfrak{F}_K)$ reduces into $H$.

**Definition.** Two subgroups $H, K$ of $G$ are termed $\mathfrak{F}$—equivalent, denoted $H \sim K$, if the set of $\mathfrak{F}$—bases of $G$ reducing into $H$ is the same as the set of $\mathfrak{F}$—bases of $G$ reducing into $K$.

**Remark.** If we take $\mathfrak{F}(p) = \{\text{the class of unit groups}\}$, for all primes $p$, then $\mathfrak{F} = \mathfrak{N}$, where $\mathfrak{N}$ is the class of finite nilpotent groups, and the above definition is just the definition due to R. Carter, of equivalency of two subgroups of $G$. (see,
Proposition 5. Let $H$ and $K$ be two subgroups of $G$. If $H \approx K$ in $G$, then $H$ is \( \mathfrak{F} \)-subnormal in \( <H, K> \).

Proof. If every \( \mathfrak{F} \)-basis of $G$ reduces into $H$, then $H$ is \( \mathfrak{F} \)-subnormal in $G$ by (2.1). Therefore $H$ is \( \mathfrak{F} \)-subnormal in \( <H, K> \) by proposition 1. Thus, if \( \mathfrak{M} \) is the set of every \( \mathfrak{F} \)-basis of $G$ reducing into $H$, we can assume that \( \mathfrak{M} \) does not contain all \( \mathfrak{F} \)-bases of $G$. Let $L$ be the stabilizer of \( \mathfrak{M} \) in $G$, i.e.,

\[
L = \{ g \in G \mid \mathfrak{M} = \mathfrak{M}^g \}.
\]

Now let \( \mathfrak{F}(\mathfrak{E}) \) be an \( \mathfrak{F} \)-basis of $G$ reducing into $H$, i.e., \( \mathfrak{F}(\mathfrak{E}) \in \mathfrak{M} \). For $h \in H$, since \( (S^p)^h \cap H_{S^p} = (S^p \cap H_{S^p})^h = (S^p \cap H_{S^p})^h = (H^p \cap H_{S^p})^h = (H^p \cap H_{S^p})^h \) for each prime $p$, where $S^p \in \mathfrak{E}$ and $H^p$ is Sylow $p$-complement of $H$, \( \mathfrak{F}(\mathfrak{E})^h = \mathfrak{F}(\mathfrak{E}^h) \) reduces into \( H = H^h \). Therefore \( \mathfrak{M} = \mathfrak{M}^h \) and hence $H$ is a subgroup of $L$. Since $H \approx K$ in $G$, \( \mathfrak{M} \) is the set of all \( \mathfrak{F} \)-bases of $G$ reducing into $K$ and hence, by the same reason as above, $K$ is a subgroup of $L$. Let \( \mathfrak{F}(\mathfrak{E}') \) be any \( \mathfrak{F} \)-basis of $G$. Then, since any two \( \mathfrak{F} \)-bases of $G$ are conjugate in $G$, there is $g$ in $G$ such that \( \mathfrak{F}(\mathfrak{E}) = \mathfrak{F}(\mathfrak{E}')^g \). Now suppose that $L = G$. Then \( \mathfrak{F}(\mathfrak{E}') \) reduces into $H$ which contradicts the hypotheses of \( \mathfrak{M} \), since $L$ is the stabilizer of \( \mathfrak{M} \). Thus $L \neq G$.

If \( \mathfrak{F}(\mathfrak{E}_L) \) is an \( \mathfrak{F} \)-basis of $L$ reducing into $H$, then, for a Sylow system $\mathfrak{E}$ of $G$ which is the extension of $\mathfrak{E}_L$, \( \mathfrak{F}(\mathfrak{E}) \) is an \( \mathfrak{F} \)-basis of $G$ reducing into $H$ by lemma 4. Since $H \approx K$ in $G$, \( \mathfrak{F}(\mathfrak{E}) \) reduces into $K$. Hence \( \mathfrak{F}(\mathfrak{E}_L) \) reduces into $K$ by lemma 4. Similary, if \( \mathfrak{F}(\mathfrak{E}_L) \) is an \( \mathfrak{F} \)-basis of $L$ reducing into $K$, then \( \mathfrak{F}(\mathfrak{E}_L) \) reduces into $H$. Therefore $H \approx K$ in $L$. We will prove the proposition by using induction on the group order. Since $|L| < |G|$, we see that, by working on $L$, $H$ is \( \mathfrak{F} \)-subnormal in \( <H, K> \).

Lemma 6. Let $H$ be a subgroup of $G$. Let $\mathfrak{M}$ be the set of all \( \mathfrak{F} \)-bases of $G$ reducing into $H$ and $L$ be the stabilizer of $\mathfrak{M}$. Then $H$ is an \( \mathfrak{F} \)-subnormal subgroup of $L$.

Proof. In the proof of above proposition, we showed that $H$ is a subgroup of $L$. Let $\mathfrak{F}(\mathfrak{E}_L)$ be any \( \mathfrak{F} \)-basis of $L$. Then there exists a Sylow system $\mathfrak{E}$ of $G$ which is an extension of the Sylow system $\mathfrak{E}_L$ of $L$. Now $\mathfrak{E}_L$ reduces into some conjugate of $H$ in $L$, say $H'$. Hence $\mathfrak{E}$ reduces into $H'$ by lemma 4. Therefore $\mathfrak{E}'$ reduces into $H$ and $\mathfrak{F}(\mathfrak{E})^g = \mathfrak{F}(\mathfrak{E}')^g$ reduces into $H$. Since $L$ is the stabilizer of $\mathfrak{M}$, $\mathfrak{F}(\mathfrak{E})$ reduces into $H$ and hence $\mathfrak{F}(\mathfrak{E}_L)$ reduces into $H$ by lemma 4. Thus $H$ is $\mathfrak{F}$-subnormal in $L$ by (2.1).

We need the following result of H. Wielandt. A subgroup $H$ of $G$ is said to be subnormal in $G$ if $H$ is $\mathfrak{F}$-subnormal in $G$ for $\mathfrak{F} = \mathfrak{M}$.

Lemma 7. [6, Theorem 6.5] If $H$ and $K$ are subnormal subgroups of $G$, then $\langle H, K \rangle$ is a subnormal subgroup of $G$. 

Definition in [4]).
PROPOSITION 8. Let $H$ be a subgroup of $G$. Let $\mathcal{W}$ be the set of all $\mathfrak{g}$-bases of $G$ reducing into $H$ and $L$ be the stabilizer in $G$ of $\mathcal{W}$. Suppose that $\mathcal{W}$ forms a block. Then if $H$ is subnormal in $L$, the equivalence class which contains $H$ has a maximal element.

PROOF. We will show that, if $H \sim K$ in $G$, then $H \sim <H,K>$ in $G$. Then it follows that $M(H;\mathfrak{g})= <K,H \sim K$ in $G$. $K$ is a subgroup of $G$ is a maximal element in the equivalence class which contains $H$.

Let $\mathfrak{a}(\mathfrak{a})$ be any $\mathfrak{g}$-basis of $G$ reducing into $<H,K>$. Since $H \sim K$ in $G$, $H$ is $\mathfrak{g}$-subnormal in $<H,K>$ by proposition 5 and hence $\mathfrak{a}(\mathfrak{a})$ reduces into $H$ by proposition 2. Therefore we need show that any $\mathfrak{g}$-basis reducing into $H$ reduces into $<H,K>$.

Let $\mathfrak{a}(\mathfrak{a})$ be any $\mathfrak{g}$-basis of $G$ reducing into $H$, i.e., $\mathfrak{a}(\mathfrak{a}) \in \mathcal{W}$. Then $\mathfrak{a}(\mathfrak{a})$ reduces into $L^g$ for some $g \in G$. Since $H$ is $\mathfrak{g}$-subnormal in $L$ by lemma 6, $H^g$ is $\mathfrak{g}$-subnormal in $L^g$. Therefore $\mathfrak{a}(\mathfrak{a})$ reduces into $H^g$ and $\mathfrak{a}(\mathfrak{a}) \in \mathcal{W} \cap \mathfrak{w}^g$. Thus, since $\mathfrak{w}$ is a block, $\mathfrak{w}=\mathfrak{w}^g$ and hence $g \in L$ and so $\mathfrak{a}(\mathfrak{a})$ reduces into $L$. Since $H$ is a subnormal subgroup of $L$, $K$ is a subnormal subgroup of $L$ and hence $<H,K>$ is subnormal in $L$ by lemma 7. Therefore $\mathfrak{a}(\mathfrak{a})$ reduces into $<H,K>$ by proposition 2.

LEMMA 9. Let $H$ be a subgroup of $G$. Suppose that the set of all $\mathfrak{g}$-bases of $G$ reducing into $H$ forms a block. Then $H \sim R(H;\mathfrak{g})$ in $G$.

PROOF. This lemma follows from the definition of $\mathfrak{g}$-equivalent and (2.5).

PROPOSITION 10. Let $H$ be a subgroup of $G$. Let $\mathcal{W}$ be the set of all $\mathfrak{g}$-bases of $G$ reducing into $H$ and $L$ be the stabilizer in $G$ of $\mathcal{W}$. Suppose that $H$ is subnormal in $L$ and $\mathcal{W}$ forms a block. Then we have that $M(H;\mathfrak{g})=R(H;\mathfrak{g})$.

PROOF. Since $\mathcal{w}$ is a block, we have $R(H;\mathfrak{g})=Q(H;\mathfrak{g})=L$ by (2.6). Now by lemma 9, $H \sim R(H;\mathfrak{g})$ in $G$, and hence we have $R(H;\mathfrak{g}) \subseteq M(H;\mathfrak{g})$ by the construction of $M(H;\mathfrak{g})$. On the other hand, since $M(H;\mathfrak{g})$ is a subgroup of $L$, it follows that $M(H;\mathfrak{g}) \subseteq R(H;\mathfrak{g})$. Therefore $M(H;\mathfrak{g})=R(H;\mathfrak{g})$.

PROPOSITION 11. Let $H$ be a subgroup of $G$. Suppose that the set of all $\mathfrak{g}$-bases of $G$ reducing into $H$ forms a block. Then we have that $N_G(R(H;\mathfrak{g}))=R(H;\mathfrak{g})$.

PROOF. By (2.8), $R(H;\mathfrak{g})$ is an $\mathfrak{g}$-subnormalizer of $H$ in $G$. Therefore $H$ is $\mathfrak{g}$-subnormal in $R(H;\mathfrak{g})$ and hence $H$ is $\mathfrak{g}$-subnormal in $N_G(R(H;\mathfrak{g}))$ since $R(H;\mathfrak{g})$ is normal in $N_G(R(H;\mathfrak{g}))$. Thus we have $N_G(R(H;\mathfrak{g})) \subseteq R(H;\mathfrak{g})$, so that $N_G(R(H;\mathfrak{g})) = R(H;\mathfrak{g})$.

PROPOSITION 12. Let $H$ be a subgroup of $G$. Suppose that the set of all $\mathfrak{g}$-bases of $G$ reducing into $H$ forms a block. Then $R(H;\mathfrak{g})$ is the least $\mathfrak{g}$-abnormal subgroup $K$ of $G$ such that every $\mathfrak{g}$-basis of $G$ reducing into $H$ reduces also into $K$.

PROOF. It follows from (2.2), (2.3) and (2.4) that $R(H;\mathfrak{g})$ is the $\mathfrak{g}$-abnormal
subgroup of $G$ such that every $\mathfrak{F}$-basis of $G$ reducing into $H$ reduces also into $R(H;\mathfrak{F})$.

Let $\mathfrak{F}(\mathfrak{S})$ be any $\mathfrak{F}$-basis of $G$ which reduces into $H$ and let $a$ be any element of $R(H;\mathfrak{F})$. Then, by lemma 9, $\mathfrak{F}(\mathfrak{S})^a$ is an $\mathfrak{F}$-basis of $G$ which reduces into $H$. Now suppose $K$ as in the theorem. Then $\mathfrak{F}(\mathfrak{S})^a$ reduces into $K$ and thus $a$ is in $R(K;\mathfrak{F})$. Therefore, $R(H;\mathfrak{F}) \subseteq R(K;\mathfrak{F})$. Hence we have $R(H;\mathfrak{F}) \subseteq K$ by (2.2) since $K$ is $\mathfrak{F}$-abnormal and the proof is complete.

**Proposition 13.** Let $H$ be a subgroup of $G$ and let $K$ be a subgroup of $G$ which contains $R(H;\mathfrak{F})$. Suppose that the set of all $\mathfrak{F}$-bases of $G$ reducing into $H$ forms a block. Then $R(H;\mathfrak{F})$ is the $\mathfrak{F}$-reducer of $H$ in $K$.

**Proof.** Let $A/B$ be an $\mathfrak{F}$-central $H$-composition factor of $G$. Then, by (2.7), $R(H;\mathfrak{F})$ covers $A/B$ and hence $K$ covers $A/B$. Now $A/B$ is isomorphic to $A \cap K/B \cap K$ as $H$-groups, so therefore $A \cap K/B \cap K$ is an $\mathfrak{F}$-central $H$-composition factor of $K$. Thus $R_k(H;\mathfrak{F})$ covers $A \cap K/B \cap K$ by (2.7), where $R_k(H;\mathfrak{F})$ denote the $\mathfrak{F}$-reducer of $H$ in $K$. Therefore $R_k(H;\mathfrak{F})$ covers $A/B$. Hence $R(H;\mathfrak{F}) \subseteq R_k(H;\mathfrak{F})$ by (2.7).

Conversely, now let $\mathfrak{F}(\mathfrak{S}_K)$ be any $\mathfrak{F}$-basis of $K$ which reduces into $H$. Then there exists a Sylow system $\mathfrak{S}$ of $G$ which is extension of $\mathfrak{S}_K$ and $\mathfrak{F}(\mathfrak{S})$ reduces into $H$ by lemma 4. Therefore $\mathfrak{F}(\mathfrak{S})$ reduces into $R(H;\mathfrak{F})$ by (2.4), and so $\mathfrak{F}(\mathfrak{S}_K)$ reduces into $R(H;\mathfrak{F})$ by lemma 4. Thus by Proposition 12, $R_k(H;\mathfrak{F}) \subseteq R(H;\mathfrak{F})$ and the proof is complete.

**References**


(Received: April 30, 1977)