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Waiting Time Distribution and Entropy (Average Uncertainty) on Left-Turning against Traffic

(PART II)

Tokuichi YONEMORI

ABSTRACT

This paper concerns further discussion\(^{1}\) of waiting time distribution and entropy (average uncertainty) experienced by left-turning\(^{2}\) cars that face traffic, and serve in order of arrival, and wait for service in one queue as long as necessary. The left-turning cars cross street as soon as a required gap between two cars develops. What would happen to the left-turning cars if the distances between successive cars are independently, identically distributed\(^{3}\), and the crossing time of the left-turning cars is constant, or exponentially distributed? In this paper, the GI!G!M model, the M!D!M model and M!M!M model are discussed\(^{4}\).

I. The GI!G!M\(^{5}\) model

We suppose that traffic is moving constant, and is not interrupted by

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\(^{1}\)This is referred to ‘Waiting Time Distribution and Entropy (Average Uncertainty) on Left-Turning against Traffic’ by T. YONEMORI (1977).

\(^{2}\)‘left-turning’ could be understood as ‘right-turning’ if people keep to the left in driving.

\(^{3}\)In this paper, we only handle the case of Poisson (Markov) arrivals of left-turning cars.

\(^{4}\)To confirm the mathematical analysis of these different models, the computer simulations are fully employed (see APPENDIX A, B and C).

\(^{5}\)‘Arbitrary recurrent (General Independent) arrivals of successive cars, Arbitrary (General) crossing time of a left-turning car, and Poisson (Markov) arrivals of left-turning cars’
the left-turning cars. Consider the arrivals of successive cars occur at
epochs \( T_1, T_2, \ldots, \) with the interevent times \( X_i = T_i - T_{i-1} \) \((i = 1, 2, 3, \ldots; T_0 = 0)\) mutually independent, identically distributed positive random
variables, with a distribution function \( P\{X_i \leq t\} = F(t)\), where \( F(t) = 0 \) for \( t < 0 \) and \( F(0) < 1 \). Also in order to cross the street, a left-
turning car needs a gap between two cars of at least \( y \) time units, where
a value \( y \) is sampled from a general crossing time distribution function
\( P\{Y \leq t\} = H(t)\), where \( H(t) = 0 \) for \( t < 0 \) and \( H(0) < 1 \), at the
time of his service. Suppose the left-turning car samples a value \( x \) from
\( F(t) \) and if \( x < y \), then he is blocked. Otherwise he crosses the street.
On the otherhand, the arrivals of the left-turning cars occur at epochs
\( \tilde{T}_1, \tilde{T}_2, \ldots, \), with the interevent times \( \tilde{X}_i = \tilde{T}_i - \tilde{T}_{i-1} \) \((i = 1, 2, 3, \ldots; \)
\( \tilde{T}_0 = 0)\) mutually independent, identically distributed positive random
variables, with distributed function \( P\{\tilde{X}_i \leq t\} = A(t)\), where
\[
A(t) = \begin{cases} 
1 - e^{-\alpha t} & (t \leq 0) \\
0 & (t < 0)
\end{cases} (1-1)
\]
and with the corresponding density function
\[
a(t) = \frac{d}{dt} A(t) = \alpha e^{-\alpha t} \quad (t \geq 0) (1-2)
\]
Note that \( \tilde{X}_i \) \((j = 1, 2, 3, \ldots)\) has negative exponential distribution with
mean \( E(\tilde{X}_i) = \alpha^{-1} \), and variance \( V(\tilde{X}_i) = \alpha^{-2} \). Now the probability that
the first success in crossing the street occurs after exactly \( n \) blockings
is given by \( (1-p)^n p \), where \( p \) is the probability of success in crossing the
street at any particular trial. If we let \( N \) be the number of blockings
preceding the first success, then \( N \) has geometric distribution
\[
P\{N = n\} = (1-p)^n p \quad (n = 0, 1, 2, \ldots) \ldots \ldots (1-3)
\]
Notice that the random variable \( N \) has mean
\[
E(N) = q / p, \quad (1-4)
\]
and variance
\[
V(N) = q / p^2, \quad (1-5)
\]
where
\[ q = 1 - p. \]  
(1-6)

Similarly define the event \( E_n \) to be the event of the first success occurs after exactly \( n \) blockings for \( n = 0, 1, 2, \ldots \). Then the probability of the event \( E_n \) is
\[ P(E_n) = (1 - p)^n p = q^n p. \]  
(1-7)

We realize that the variables \( X_i \) (\( i = 1, 2, 3, \ldots \)) are independent of one another and their distribution is independent of \( x \) as well as \( n \). Also the variables \( \tilde{X}_i \) are independent of one another as well as \( X_i \). Now we get
\[ p = P\{y < x\} = \int_{x=0}^{\infty} [1 - F(x)] dH(x). \]  
(1-8)

Clearly \( q = 1 - p \).

Now define \( F^*(t) \) to be the conditional distribution of \( x \), and \( H^*(t) \) to be the conditional distribution of \( y \), we get
\[ F^*(t) = \frac{P\{x < y, x < t\}}{P\{x < y\}} = \frac{1}{q} \int_{x=0}^{t} [1 - H(x)] dF(x). \]  
(1-9)

and
\[ H^*(t) = \frac{P\{y \leq x, y < t\}}{P\{y \leq x\}} = \frac{1}{p} \int_{y=0}^{t} [1 - F(y)] dH(y). \]  
(1-10)

Denote by \( \gamma(s) \) the Laplace-Stieltjes transform of the conditional distribution function \( F^*(t) \),
\[ \gamma(s) = \int_{t=0}^{\infty} e^{-st} dF^*(t) \quad (\text{Re } s \geq 0). \]  
(1-11)

And also denote \( \eta(s) \) the Laplace-Stieltjes transform of the conditional distribution function \( H^*(t) \),
\[ \eta(s) = \int_{t=0}^{\infty} e^{-st} \, dH^*(t). \] (1-12)

Similarly let \( \tilde{\omega}_n(s) \) to be the Laplace-Stieltjes transform of \( \tilde{W}_n(t) \), where \( \tilde{W}_n(t) \) is the service time distribution function in the event \( E_n \). Then by the convolution theorem for transforms,

\[ \tilde{\omega}_n(s) = \gamma^n(s) \eta(s). \] (1-13)

Now define \( \tilde{W}(t) \) to be the (total) service time distribution function of a left-turning car, we have

\[ \tilde{W}(t) = \sum_{n=0}^{\infty} P(E_n) \tilde{W}_n(t). \] (1-14)

Hence by the linearity property of transforms

\[ \tilde{\omega}(s) = \sum_{n=0}^{\infty} P(E_n) \tilde{\omega}_n(t). \] (1-15)

We now let \( N_k^* \) be the number of left-turning cars in the system (including any in service, but excluding the departing left-turning car) at the instant the kth left-turning car completes service. That is, if \( T_1, T_2, \ldots \), are successive service completion points, then \( N_k^* \) is the state of the system at \( T_k + 0 \). Then by the law of total probability,

\[ P\{N_{k+1} = j \} = \sum_{i=0}^{\infty} P\{N_{k+1} = j \mid N_k^* = i \} P\{N_k^* = i \} \] (1-16)

\[ (j = 0, 1, 2, \ldots; k = 1, 2, \ldots), \]

During any interval of length \( t \), we have

\[ P\{N_{k+1} = j \mid N_k^* = 0 \} = \int_{t=0}^{\infty} \frac{(\alpha t)^j}{j!} \, e^{-\alpha t} \, d\tilde{W}(t) \quad (j \geq 0), \] (1-17)

and

\[ P\{N_{k+1} = j \mid N_k^* = i \} = \int_{t=0}^{\infty} \frac{(\alpha t)^{j-i+1}}{(j-i+1)!} \, e^{-\alpha t} \, d\tilde{W}(t) \] (1-18)

\[ (i > 0, j \geq i-1), \]
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and

\[ P \{ N_{k+1} = j \mid N_k = i \} = 0 \quad (i > 0, j < i - 1). \quad (1-19) \]

Let us define

\[ \tilde{P}_j = \int_{t=0}^{\infty} \frac{(\alpha t)^j}{j!} e^{-\alpha t} d\tilde{W}(t) \quad (j = 0, 1, 2, \ldots), \quad (1-20) \]

Then the transition probabilities are

\[ P \{ N_{k+1} = j \mid N_k = i \} = \begin{cases} \tilde{P}_i & \text{if } i = 0 \\ \tilde{P}_{j-i+1} & \text{if } i > 0 \text{ and } j \geq i - 1 \\ 0 & \text{if } i > 0 \text{ and } j < i - 1 \end{cases} \quad (1-21) \]

Note that the transition probabilities are independent of the value of the index k, that is, are the same for every pair of successive left-turning cars. We wish to know the distribution of the number of left-turning cars left behind by an arbitrary departing car. Let \( \Pi^*_j \) be the probability that an arbitrary departing car leaves behind \( j \) other left-turning cars in the system, that has been operating for a sufficiently long period of time. From a mathematical point of view, it can be shown\(^6\) using the theory of Markov chains that a unique proper stationary distribution

\[ \Pi^*_j = \lim_{k \to \infty} P \{ N_k = j \} \quad (j = 0, 1, 2, \ldots) \quad (1-22) \]

independent of the initial conditions, exist if and only if the offered load \( \rho < 1. \) (if \( \rho \geq 1 \), then \( \Pi^*_j = 0 \) for all finite \( j \).)

Equation (1—16) becomes

\[ \Pi^*_j = \tilde{p}_j \Pi^*_0 + \sum_{i=1}^{j+1} \tilde{p}_{j-i+1} \Pi^*_i \quad (j = 0, 1, 2, \ldots). \quad (1-23) \]

Then the normalization equation

\[ \sum_{j=0}^{\infty} \Pi^*_j = 1, \quad (1-24) \]

\(^6\) See 'Introduction to Queueing Theory' (pg.170-171) by R. B. Cooper (1972).
and equation (1—23) gives us distribution \( \{ \Pi_j^* \} \) such that

\[
\Pi_j^* = \begin{cases} 
1 - \rho & \text{if } j = 0 \\
(1 - p_1^0) \prod_{i=0}^{j-1} \Pi_i^* - p_1^0 & \text{if } j = 1 \\
(1 - p_1^j) \prod_{i=0}^{j-1} - \prod_{i=0}^{j-1} p_1 & \text{if } j = 2 \\
(1 - p_1^j) \prod_{i=0}^{j-1} - \prod_{i=0}^{j-1} p_1 & \text{if } j \geq 3
\end{cases}
\] (1—25)

where

\[
\rho = a \left[ (-1) \frac{d}{ds} \tilde{\omega}(s) \big|_{s=0} \right] < 1
\] (1—26)

if we let

\[
g(z) = \sum_{j=0}^{\infty} \Pi_j^* z^j
\] (1—27)

be the equilibrium probability generating function of the number of left-turning cars left behind in the system at the instant of an arbitrary left-turning car completes service and departs from the system, we get using equation (1—23)

\[
g(z) = \sum_{j=0}^{\infty} (p_j^0 \prod_{i=0}^{j} + \sum_{i=1}^{j+1} p_{j-i+1}^j \prod_{i=0}^{j-i} z^j.
\] (1—28)

If we also define

\[
\psi(z) = \sum_{j=0}^{\infty} p_j^z j^z,
\] (1—29)

we obtain from (1—28), using (1—25),

\[
g(z) = \frac{\psi(z)(z-1)}{z-\psi(z)} (1 - \rho).
\] (1—30)

Since \( \tilde{\omega}(s) \) the Laplace-Stieljes transform of \( \tilde{W}(t) \), then equation (1—15) could be also expressed by

\[
\tilde{\omega}(s) = \int_{t=0}^{\infty} e^{-st} d\tilde{W}(t).
\] (1—31)

Then the substitution of (1—20) into (1—29) yields

\[
\psi(z) = \tilde{\omega}(\alpha - \alpha z).
\] (1—32)
Therefore, equation (1—30) can be written
\[ g(z) = \frac{\omega(\alpha - \alpha z)(z - 1)}{z - \omega(\alpha - \alpha z)^{\prime}} (1 - \rho) . \] (1—33)

Observing
\[ \tau_n = \int_{t=0}^{\infty} t^n d\tilde{W}(t) = (-1)^n \frac{d^n}{ds^n} \tilde{\omega}(s) \bigg|_{s=0}, \] (1—34)
we denote
\[ \phi^{(1)}(1) = \frac{d}{dz} \tilde{\omega}(\alpha - \alpha z) \big|_{z=1} = \alpha \int_{t=0}^{\infty} t d\tilde{W}(t) \]
\[ = \alpha \left[ (-1)^1 \frac{d}{ds} \tilde{\omega}(s) \big|_{s=0} \right] = \alpha \tau_1 = \rho , \] (1—35)
and
\[ \phi^{(2)}(1) = \frac{d^2}{dz^2} \tilde{\omega}(\alpha - \alpha z) \big|_{z=1} = \alpha^2 \int_{t=0}^{\infty} t^2 d\tilde{W}(t) \]
\[ = \alpha^2 \left[ (-1)^2 \frac{d^2}{ds^2} \tilde{\omega}(s) \big|_{s=0} \right] = \alpha^2 \tau_2 , \] (1—36)
and
\[ \phi^{(3)}(1) = \frac{d^3}{dz^3} \tilde{\omega}(\alpha - \alpha z) \big|_{z=1} = \alpha^3 \int_{t=0}^{\infty} t^3 d\tilde{W}(t) \]
\[ = \alpha^3 \left[ (-1)^3 \frac{d^3}{ds^3} \tilde{\omega}(s) \big|_{s=0} \right] = \alpha^3 \tau_3 . \] (1—37)

Now if we let \( g^{(1)}(1) = \frac{d}{dz} g(z) \big|_{z=1}, \ g^{(2)}(1) = \frac{d^2}{dz^2} g(z) \big|_{z=1}, \)
and \( \phi^{(1)}(1) = \rho , \) we get, after some manipulation of (1—33),
\[ g^{(1)}(1) = \rho + \frac{\phi^{(1)}(1)}{2(1 - \rho)} = \rho + \frac{\alpha^2 \tau_2}{2(1 - \rho)} , \] (1—38)
and

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Here notice that the mean \( E(N^*) \) can be obtained from the probability generating function as
\[
E(N^*) = g'(1)
\]
and the variance \( V(N^*) \) as
\[
V(N^*) = g''(1) + g'(1) - [g'(1)]^2.
\]

We turn now to the waiting time distribution function in the queue \( W_q(t) = P\{W_q \leq t\} \). Let \( W_q(t) \) have the Laplace-Stieltjes transform \( \omega_q(s) \),
\[
\omega_q(s) = \int_0^\infty e^{-st} \, dW_q(t),
\]
and denote by \( \omega(s) \) the Laplace-Stieltjes transform of the distribution function of the sojourn time (the sum of the waiting time in the queue and service time) of a left-turning car. Since the waiting time in the queue and service time are independent, we clearly get
\[
\omega(s) = \omega_q(s) \bar{\omega}(s).
\]

Since service is in arrival order, the left-turning cars left behind by the departing car must all have arrived during his sojourn time. Let \( W(t) = P\{W \leq t\} \) to be the total waiting time (sojourn time) distribution of a left-turning car in the system, and let \( p_j \) be the probability that \( j \) left-turning cars arrive during his sojourn time, we get
\[
p_j = \int_t^\infty \frac{(\alpha t)^j}{j!} \, e^{-\alpha t} \, dW(t).
\]
Since
\[
g(z) = \sum_{j=0}^{\infty} p_j z^j,
\]
we get, after substitution of (1−44) into (1−45),
\[ g(z) = \omega(a - az). \] (1−46)

Using equations (1−46) and (1−33) in (1−43), we have widely known Pollaczek-Khintchin formula
\[ \omega_q(s) = \frac{s(1-p)}{s - \alpha(1-\bar{\omega}(s))}, \] (1−47)
and also
\[ \omega(s) = \frac{s(1-p)\bar{\omega}(s)}{s - \alpha(1-\bar{\omega}(s))}. \] (1−48)

To calculate the mean waiting time in the queue \( E(W_q) \), note
\[ E(W_q) = (-1)^1 \frac{d}{ds} \omega_q(s) \bigg|_{s=0}. \] (1−49)

And since
\[ E(W_q^2) = (-1)^2 \frac{d^2}{ds^2} \omega_q(s) \bigg|_{s=0}, \] (1−50)
the variance \( V(W_q) \) could be obtained from
\[ V(W_q) = E(W_q^2) - E^2(W_q). \] (1−51)

It is also true for the mean total waiting time (sojourn time) in the system. Then we get
\[ E(W) = (-1)^1 \frac{d}{ds} \omega(s) \bigg|_{s=0}, \] (1−52)
\[ E(W^2) = (-1)^2 \frac{d^2}{ds^2} \omega(s) \bigg|_{s=0}. \] (1−53)

Clearly
\[ V(W) = E(W^2) - E^2(W). \] (1−54)
Since the term \( r_n \) is given by equation (1—34), then equations (1—49) and (1—51) gives us

\[
E(W_q) = \frac{\alpha \tau_2}{2(1 - \rho)}, \quad (1-55)
\]

and

\[
V(W_q) = \frac{\alpha^2 \tau_2^2}{4(1 - \rho)^2} + \frac{\alpha \tau_3}{3(1 - \rho)}. \quad (1-56)
\]

It also follows from equation (1—52) and (1—54) that

\[
E(W) = \tau_1 + \frac{\tau_2}{2(1 - \rho)} \quad (1-57)
\]

and

\[
V(W) = (\tau_2 - \tau_1^2) + \frac{\alpha^2 \tau_2^2}{4(1 - \rho)^2} + \frac{\alpha \tau_3}{3(1 - \rho)}. \quad (1-58)
\]

We now proceed to write an equation for the entropy (average uncertainty) experienced by the left-turning cars in the system. In the previous paper (1977), we have shown that the entropy \( \nu \) experienced by a left-turning car is given by

\[
\nu = H(P(E_0), P(E_1), \ldots) = E(N) I_s + I_a, \quad (1-59)
\]

where \( I_s \), the amount of sureness-in-crossing-this-time, should be measured by

\[
I_s = \log_2(q^{-1}), \quad (1-60)
\]

and \( I_a \), the amount of anxiety in crossing,

\[
I_a = \log_2(p^{-1}). \quad (1-61)
\]

Note that \( \Pi_j^* \) is the probability that an arbitrary departing left-turning car leaves \( j \) other left-turning cars in the system, and also services are in order of arrivals. Consider first the case that the departing car leaves
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no other left-turning cars behind. Then there exists only \( \nu \) experienced by the departing car. In the case that the departing car leaves one left-turning car, there exist \( \nu \) by the departing car, and \( 2 \nu \) by the left-turning car left behind. And in the case of two left, there exist \( \nu \) by the departing car, \( 2 \nu \) by the first arrived car, and \( 3 \nu \) by the second arrived car. In general, in the case of fixed \( j \), there exist \( \nu \) by the departing car, \( 2 \nu \) by the first arrived car, \( 3 \nu \) by the second arrived car, \( \ldots \), and \( (j+1) \nu \) by the (last) \( j \)th arrived car. If we define \( \nu_j \) to be the entropy in the case of \( j \) left-turning cars left behind in the system, \( \nu_j \) is given by

\[
\nu_j = \sum_{k=1}^{j+1} k \nu = \nu \sum_{k=1}^{j+1} k \quad (j \geq 0). \quad (1-62)
\]

Therefore the entropy \( \bar{\nu} \) experienced by the system could be obtained from, by the law of total probability,

\[
\bar{\nu} = \sum_{j=0}^{\infty} \Pi_j \nu_j = \nu \sum_{j=0}^{\infty} \Pi_j \left[ \frac{(j+1)(j+2)}{2} \right]
\]

\[
= \frac{\nu}{2} \left[ \sum_{j=0}^{\infty} j^2 \Pi_j + 3 \sum_{j=0}^{\infty} j \Pi_j + 2 \sum_{j=0}^{\infty} \Pi_j \right]
\]

\[
= \frac{\nu}{2} \left[ \text{V}(N^*) + E^2(N^*) + 3 E(N^*) + 2 \right]. \quad (1-63)
\]

II The M!D!M\(^{(7)}\) model

Consider the following process. Traffic is moving constant, and the arrivals of successive cars occur at epochs \( T_1, T_2, \ldots \), with the interevent times \( X_i = T_i - T_{i-1} \) (\( i = 1, 2, \ldots; T_0 = 0 \)) mutually independent, identically distributed positive random variables, with \( P \{ X_i \leq t \} = F(t) \), where

\( ^{(7)} \) ‘Poisson (Markov) arrivals of successive cars, Constant (Deterministic) crossing time of left-turning cars, Poisson (Markov) arrivals of left-turning cars’
\[
F(t) = \begin{cases} 
1 - e^{-\lambda t} & (t \geq 0) \\
0 & (t < 0)
\end{cases} \tag{2-1}
\]

with
\[
f(t) = \frac{d}{dt} F(t) = \lambda e^{-\lambda t} \quad (t \geq 0). \tag{2-2}
\]

And also the crossing time is constant such that the distribution function
\[P \{ Y \leq t \} = H(t)\]
is given by
\[
H(t) = \begin{cases} 
1 & (t \geq t_0) \\
0 & (t < t_0)
\end{cases} \tag{2-3}
\]
and the arrival distribution function of the left-turning cars is described
in equation (1—1). Substituting equation (2—1) and (2—3) into (1—8),
we get the probability of no blocking (or success) on left-turning such
that
\[
p = \int_{x=0}^{\infty} H(x) \, dF(x) = e^{-\lambda t_0} \tag{2-4}
\]
and
\[
q = 1 - p = 1 - e^{-\lambda t_0} \tag{2-5}
\]
Now from (1—11), we get
\[
\gamma(s) = \frac{\lambda}{s + \lambda} \frac{1 - e^{-(s+\lambda)t_0}}{1 - e^{-\lambda t_0}}, \tag{2-6}
\]
and from (1—12), we get
\[
\eta(s) = \frac{e^{-(s+\lambda)t_0}}{e^{-\lambda t_0}}, \tag{2-7}
\]
Since, from (1—13),
\[
\tilde{\omega}_n(s) = \left[ \frac{\lambda}{s + \lambda} \frac{1 - e^{-(s+\lambda)t_0}}{1 - e^{-\lambda t_0}} \right]^n \left[ \frac{e^{-(s+\lambda)t_0}}{e^{-\lambda t_0}} \right], \tag{2-8}
\]

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and, from (1—7),
\[ P(E_n) = (1 - e^{-\lambda t_0})^n e^{-\lambda t_0}, \]
we get, from (1—15),
\[ \tilde{\omega}(s) = \frac{(s + \lambda) e^{-(s+\lambda)t_0}}{s + \lambda e^{-(s+\lambda)t_0}}. \]

Noting that we let \( N^* \) be the random variable of the number of left-turning cars in the system (including any in service, but excluding the departing left-turning car), and also \( \Pi^*_j \) be the probability that the departing car leaves \( j \) other left-turning cars behind in the system, we obtain from (1—34)
\[ \tau_1 = (-1)^1 \frac{d}{ds} \tilde{\omega}(s) \bigg|_{s=0} = \frac{e^{\lambda t_0} - 1}{\lambda}, \]
and
\[ \tau_2 = (-1)^2 \frac{d^2}{ds^2} \tilde{\omega}(s) \bigg|_{s=0} = \frac{2 e^{2\lambda t_0} - 2 \lambda t_0 e^{\lambda t_0} - 2 e^{\lambda t_0}}{\lambda^2}. \]
and
\[ \tau_3 = (-1)^3 \frac{d^3}{ds^3} \tilde{\omega}(s) \bigg|_{s=0} = \frac{3 e^{\lambda t_0} (2 \lambda t_0 + \lambda^2 t_0^2 - 2 e^{\lambda t_0} + 2 e^{2\lambda t_0} - 4 \lambda t_0 e^{\lambda t_0})}{\lambda^3}. \]

We have shown that the equilibrium probability generating function of the number of left-turning cars left behind in the system by the departing car is
\[ g(z) = \sum_{j=0}^{\infty} \prod_{j} z^{i} \]
\[ = \frac{\omega(\alpha - az)(z - 1)}{z - \omega(\alpha - az)} (1 - \rho). \] (2-14)

Then if we let \( g'(1) = \frac{d}{ds} g(z) \mid_{z=1} \), \( g''(1) = \frac{d^{2}}{ds^{2}} g(z) \mid_{z=1} \), and \( \alpha \tau_{1} = \rho \), we get
\[ g'(1) = \rho + \frac{\alpha^{2} \tau_{2}}{2(1 - \rho)} \] (2-15)

and
\[ g''(1) = \frac{\alpha^{2} \tau_{2}}{1 - \rho} + \frac{\alpha^{4} \tau_{2}^{2}}{2(1 - \rho)^{2}} + \frac{\alpha^{3} \tau_{3}}{3(1 - \rho)}. \] (2-16)

Therefore the mean \( E(N^{*}) \) and the variance \( V(N^{*}) \) are easily obtained from
\[ E(N^{*}) = g'(1) = \rho + \frac{\alpha^{2} \tau_{2}}{2(1 - \rho)} \] (2-17)

and
\[ V(N^{*}) = g''(1) + g'(1) - [g'(1)]^{2} \]
\[ = \frac{\alpha^{2} \tau_{2}}{1 - \rho} + \frac{\alpha^{4} \tau_{2}^{2}}{2(1 - \rho)^{2}} + \frac{\alpha^{3} \tau_{3}}{3(1 - \rho)} \]
\[ + \rho + \frac{\alpha^{2} \tau_{2}}{2(1 - \rho)} - \left( \rho + \frac{\alpha^{2} \tau_{2}}{2(1 - \rho)} \right)^{2}. \] (2-18)

Furthermore we have shown the relationships between \( \omega(s) \), \( \omega_{q}(s) \) and \( \omega(s) \) as
\[ \omega(s) = \omega_{q}(s) \omega(s), \] (2-19)
Waiting Time Distribution and Entropy (Average Uncertainty) on Left-Turning against Traffic [Part II] (Tokuichi Yonemori)

\[
\omega_q(s) = \frac{s(1-\rho)}{s - \alpha(1 - \tilde{\omega}(s))} \quad (2-20)
\]

\[
\omega(s) = \frac{s(1-\rho)}{s - \alpha(1 - \tilde{\omega}(s))} \quad (2-21)
\]

[See equations (1-43), (1-47), and (1-48).]

Some manipulation of (2-20) yields

\[
E(W_q) = (-1)^1 \frac{d}{ds} \omega_q(s) \bigg|_{s=0} = \frac{\alpha \tau_2}{2(1-\rho)} \quad (2-22)
\]

and

\[
V(W_q) = (-1)^2 \frac{d^2}{ds^2} \omega_q(s) \bigg|_{s=0} - E^2(W_q)
= \frac{\alpha^2 \tau_2^2}{4(1-\rho)^2} + \frac{\alpha \tau_3}{3(1-\rho)} \quad (2-23)
\]

In addition we have

\[
E(W) = (-1)^1 \frac{d}{ds} \omega(s) \bigg|_{s=0} = \tau_1 + \frac{\alpha \tau_2}{2(1-\rho)} \quad (2-24)
\]

and

\[
V(W) = (-1)^2 \frac{d^2}{ds^2} \omega(s) \bigg|_{s=0} - E^2(W)
= (\tau_2 - \tau_1^2) + \frac{\alpha^2 \tau_2^2}{4(1-\rho)^2} + \frac{\alpha \tau_3}{3(1-\rho)} \quad (2-25)
\]

Noting that, from equations (2-4), (2-5) and (1-4),

\[
E(N) = q/p = e^{\lambda t_o} - 1 \quad (2-26)
\]

and from (1-60) and (1-61),

\[
I_s = \log_2(q^{-1}) = \log_2[(1 - e^{-\lambda t_o})^{-1}] \quad (2-27)
\]

and

\[
I_a = \log_2(p^{-1}) = \log_2(e^{\lambda t_o}) \quad (2-28)
\]
we get the entropy (average uncertainty) experienced by a left-turning car
\[
\nu = H(P(E_0), P(E_1), P(E_2), \ldots)
= E(N) I_s + I_s
= (e^{4t_0} - 1) \log_2 \left[ \left( 1 - e^{-4t_0} \right)^{-1} \right] + \log_2 (e^{4t_0}).
\]  
(2-29)

Then the entropy experienced by the system is given by, from equation (1-63),
\[
\bar{\nu} = \frac{\nu}{2} \left[ V(N^*) + E^2(N^*) + 3E(N^*) + 2 \right],
\]  
(2-30)

where \(\nu\) is obtained from (2-29), \(V(N^*)\) from (2-18), and \(E(N^*)\) from (2-17).

The M!M!M\(^{(8)}\) model

Consider also the following process. Traffic flow is described in equation (2—1) in the M!D!M model. Here the crossing time distribution function is \(P\{Y \leq t\} = H(t)\), where
\[
H(t) = \begin{cases} 
1 - e^{-\mu t} & (t \geq 0) \\
0 & (t < 0)
\end{cases}
\]  
(3-1)

with
\[
h(t) = \frac{d}{dt} H(t) = \mu e^{-\mu t} \quad (t \geq 0).
\]  
(3-2)

\(^{(8)}\)Poisson (Markov) arrivals of successive cars, Exponential (Markov) crossing time of a left-turning car, Poisson (Markov) arrivals of left-turning cars
Waiting Time Distribution and Entropy (Average Uncertainty) on Left-Turning against Traffic [Part II] (Tokuichi Yonemori)

The arrival distribution function of the left-turning cars is given in equation (1-1). Substitution (2-1) and (3-1) into (1-8) we get the probability of no blocking (or success) on left-turning such that

\[
p = \int_{x=0}^{\infty} H(x) \, dF(x) = \int_{x=0}^{\infty} (1 - e^{-\mu x}) \lambda e^{-\lambda x} \, dx = \frac{\mu}{\lambda + \mu},
\]

and \( q \) the probability of blocking is

\[
q = 1 - p = \frac{\lambda}{\lambda + \mu}.
\]

Now from equation (1—11), we get

\[
\gamma(s) = \frac{\lambda + \mu}{s + \lambda + \mu},
\]

and also from equation (1—12), we get

\[
\eta(s) = \frac{\lambda + \mu}{s + \lambda + \mu}.
\]

Since from (1—13),

\[
\bar{\omega}(s) = \left( \frac{\lambda + \mu}{s + \lambda + \mu} \right)^n \left( \frac{\lambda + \mu}{s + \lambda + \mu} \right),
\]

and from (1—7),

\[
P(E_n) = \left( \frac{\lambda}{\lambda + \mu} \right)^n \left( \frac{\mu}{\lambda + \mu} \right),
\]

we get from (1—15)

\[
\bar{\omega}(s) = \frac{\mu}{s + \mu}.
\]

Then we obtain from equation (1—34) that

\[
\tau_1 = (-1)^1 \frac{d}{ds} \bar{\omega}(s) \bigg|_{s=0} = -\frac{-\mu}{(s + \mu)^2} \bigg|_{s=0} = \frac{1}{\mu},
\]

\[
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\]
and
\[ \tau_2 = (-1)^2 \frac{d^2}{ds^2} \mathcal{W}(s) \bigg|_{s=0} = \frac{2\mu}{(s + \mu)^3} \bigg|_{s=0} = \frac{2}{\mu^2}, \quad (3-11) \]
and
\[ \tau_3 = (-1)^3 \frac{d^3}{ds^3} \mathcal{W}(s) \bigg|_{s=0} = \frac{6\mu}{(s + \mu)^4} \bigg|_{s=0} = \frac{6}{\mu^3}. \quad (3-12) \]

We have shown that the equilibrium probability generating function of the number of left-turning cars left behind by the departing car in the system is
\[ g(z) = \frac{\mathcal{W}(\alpha - \alpha z)(z - 1)}{z - \mathcal{W}(-\alpha)} (1 - \rho). \quad (3-13) \]

Then if we let \( g'(1) = \frac{d}{dz} g(z) \bigg|_{z=1}, \ g''(1) = \frac{d^2}{dz^2} g(z) \bigg|_{z=1} \)
and \( \rho = \frac{\alpha}{\mu}, \) we get from (1-38),
\[ g'(1) = \rho + \frac{2\rho^2}{2(1-\rho)} = \frac{\rho}{1-\rho} \quad (3-14) \]
and from (1-39)
\[ g''(1) = \frac{2\rho^2}{1-\rho} + \frac{4\rho^4}{2(1-\rho)^2} + \frac{6\rho^3}{3(1-\rho)} \]
\[ = \frac{2\rho^2}{(1-\rho)^2}. \quad (3-15) \]

Therefore using (3-14) and (3-15), the mean \( E(N^*) \) is given by
\[ E(N^*) = g'(1) = \frac{\rho}{1-\rho}, \quad (3-16) \]
and the variance \( V(N^*) \) is given by
Waiting Time Distribution and Entropy (Average Uncertainty) on Left-Turning against Traffic [Part II] (Tokuichi Yonemori)

\[ V(N^*) = g''(1) + g'(1) - \left[ g'(1) \right]^2 \]

\[ = \frac{2 \rho^2}{(1 - \rho)^2} + \frac{\rho}{1 - \rho} - \frac{\rho^2}{(1 - \rho)^2} \]

\[ = \frac{\rho}{(1 - \rho)^2} . \quad (3-17) \]

Furthermore we have shown the relationships between \( \bar{\omega}(s) \), \( \omega_q(s) \) and \( \omega(s) \) as, from equations (1-43), (1-47) and (1-48),

\[ \omega(s) = \omega_q(s) \bar{\omega}(s), \quad (3-18) \]

\[ \omega_q(s) = \frac{s(1 - \rho)}{s - \alpha(1 - \bar{\omega}(s))} , \quad (3-19) \]

and

\[ \omega(s) = \frac{s(1 - \rho) \bar{\omega}(s)}{s - \alpha(1 - \bar{\omega}(s))} . \quad (3-20) \]

Therefore some manipulation of (3-19) yields

\[ E(W_q) = (-1)^1 \frac{d}{ds} \omega_q(s) \bigg|_{s=0} = \frac{\alpha \tau_2}{2(1 - \rho)} \]

\[ = \frac{\rho}{\mu(1 - \rho)} , \quad (3-21) \]

and

\[ V(W_q) = (-1)^2 \frac{d^2}{ds^2} \omega_q(s) \bigg|_{s=0} = E^2(W_q) \]

\[ = \frac{1}{\mu^2} \left[ \frac{2\rho - \rho^2}{(1 - \rho)^2} \right] . \quad (3-22) \]

In addition we have, from equation (1-57),

\[ E(W) = (-1)^1 \frac{d}{ds} \omega(s) \bigg|_{s=0} = \frac{1}{\mu(1 - \rho)} , \quad (3-23) \]

and from (1-58),
\[ V(W) = (-1)^2 \frac{d^2}{ds^2} \omega(s) \bigg|_{s=0} - E^2(W) = \frac{1}{\mu^2 (1 - \rho)^2} \]  

(3-24)

It is immediately evident from (3-23) and (3-24) that the total waiting time (sojourn time) distribution \( W(t) = P\{W \leq t\} \) is exponentially distributed with mean \( 1/(\mu (1 - \rho)) \); that is,

\[ W(t) = 1 - e^{-\mu(1-\rho)t} \quad (t \geq 0). \]  

(3-25)

This is what we expected on intuitive grounds.

Also from (1-20),

\[ \bar{p}_j = \left( \frac{\alpha}{\alpha + \mu} \right)^j \left( \frac{\mu}{\alpha + \mu} \right) \quad (j \geq 0), \]  

(3-26)

then we get, from equation (1-25),

\[ \Pi^*_{ij} = \begin{cases} 1 - \rho = 1 - \alpha/\mu & \text{if } j = 0 \\ (1 - \rho)^j \rho = \left(1 - \alpha/\mu\right)^j \left(\alpha/\mu\right) & \text{if } j \geq 1. \end{cases} \]  

(3-27)

Noting that, from (3-3), (3-4) and (1-4),

\[ E(N) = \frac{\lambda}{\mu}, \]  

(3-28)

and from (1-60) we get \( I_s \), the amount of sureness-in-crossing-this-time, such that

\[ I_s = \log_2 (q^{-1}) = \log_2 \left( \frac{\lambda + \mu}{\lambda} \right), \]  

(3-29)

and from (1-61) we get \( I_a \), the amount of anxiety in crossing,

\[ I_a = \log_2 (p^{-1}) = \log_2 \left( \frac{\lambda + \mu}{\mu} \right). \]  

(3-30)

Clearly the entropy (average u-certainty) experienced by a left-turning car
\[ \nu = H(P(E_0), P(E_1), P(E_2), \ldots) \]

\[ = E(N) I_s + I_a \]

\[ = \frac{\lambda}{\mu} \log_2 \left( \frac{\lambda + \mu}{\lambda} \right) + \log_2 \left( \frac{\lambda + \mu}{\mu} \right) \]  

(3-31)

implies the entropy by the system such that

\[ \tilde{\nu} = \frac{\nu}{2} \left[ V(N^*) + E^2(N^*) + 3 E(N^*) + 2 \right] = \frac{\nu}{(1 - \rho)^2} \]

\[ = \frac{\frac{\lambda}{\mu} \log_2 \left( \frac{\lambda + \mu}{\lambda} \right) + \log_2 \left( \frac{\lambda + \mu}{\mu} \right)}{(1 - \rho)^2} . \]  

(3-32)
REFERENCES


APPENDIX A

EXPLANATION of the VARIABLES

start

set zeros to initial conditions
LEFT, NTRIAL, NQUE, LEFTO, NX, NY, NZ, IND, QUE, QUE2, TX, TY, TZ, TWT

give arbitrary XX, YY, ZZ, N

write XX, YY, ZZ, N

generate X, Y, and Z using XX, YY, and ZZ

determine if the left-turning test car passes the street

Z ≥ X > Y or X > Z > Y

yes

he finally passes street

no

NQUE ≥ 1

yes

call subroutine
SUB1(1, NQUE, LEFTO, LEFT, NTRIAL, QUE, QUE2)

tno

TWT = TWT + Y

A

C

call subroutine
SUB1(2, NQUE, LEFTO, LEFT, NTRIAL, QUE, QUE2)

tno

TWT = TWT + Y

A

E

APPENDIX A

E

X > Y > Z ?

yes

while he is passing street another left-turning car joins queue

TWT = TWT + Z
NQUE = NQUE + 1
X = X - Z
Y = Y - Z

no

call subroutine
SUB2(IND, Z, ZZ, TZ, NZ, 1)

B

since he cannot pass, he waits for another trial

TWT = TWT + X

yes

generate X, Y, and Z

no

Z ≥ Y ≥ X or Y ≥ Z > X

no

X ≥ Y > Z ?

yes

when he begins next trial, he sees another left-turning car joins queue

TWT = TWT + Z
X = X - Z
NQUE = NQUE + 1
Y = Y - Z

no

Y ≥ X ≥ Z ?

yes

TWT = TWT + X
NQUE = NQUE + 1

no

Y ≥ X, X = Z ?

yes

generate X, Y, and Z

no

X > Y, Y = Z ?

yes

NQUE = NQUE + 1

F

C
write "ERROR"

stop

check if the number of trials is equal to \(N\)

\[ A \quad N_{\text{TRIAL}} \geq N \]

no \( \rightarrow \) D

yes

\[ XX = \text{FLOAT}(N_X)/T_X \]
\[ YY = \text{FLOAT}(N_Y)/T_Y \]
\[ ZZ = \text{FLOAT}(N_Z)/T_Z \]

we get the exact values of \(XX, YY, \) and \(ZZ\) in this simulation

write \(XX, YY, ZZ\)

obtain theoretical results for \(W, PBLK, \) and \(ROW\)

\[ \text{ROW} \geq 1 \]

yes \( \rightarrow \) write "OVER FLOW"

stop

no

obtain theoretical results for \(EQ, HEIKIN, \) and \(VARY\)

write \(EQ, HEIKIN, VARY\)

G
APPENDIX A

G

get the simulation results for T, TV, AWT

write T, TV, AWT

compare P(0), P(1), ...... with PAI(0), PAI(1), ......

stop
Waiting Time Distribution and Entropy (Average Uncertainty) on Left-Turning against Traffic [Part II] (Tokuichi Yonemori)

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ROW : offered load $\rho$ (for detail discussion see equation (1–26))

EQ : theoretical number of left-turning cars joining the queue from the instant an arbitrary left-turning car faces traffic till he completely crosses street (see equations (1–40), (2–17) and (3–16))

HEIKIN : theoretical mean number of left-turning cars joining the queue from the instant that an arbitrary left-turning car join the queue till he completely crosses street (see equations (1–40), (2–17) and (3–16))

VARY : theoretical variance of number of left-turning cars joining the queue from the instant that an arbitrary left-turning car joins the queue till completely crosses street (see equations (1–41), (2–18) and (3–17))

T : simulation result for the mean number of left-turning cars left behind by an arbitrary test car (compare with HEIKIN)

TV : simulation result for the variance of number of left-turning cars left behind by an arbitrary test car (compare with VARY)

AWT : simulation result for mean service time from the instant that an arbitrary left-turning car faces traffic till he completely crosses street (compare with W)

PAI(k) : theoretical probabilities that the departing left-turning car leaves k other left-turning cars behind in the system

P(k) : simulation result for the probabilities that the departing left-turning cars behind in the system

LEFT(k) : number of cases that k left-turning cars left behind in the system by an arbitrary test car

NTRIAL : number of trials in the simulation such that $0 \leq \text{NTRIAL} \leq \text{N}$

NQUE : number of left-turning cars in the queue excluding the test car

LEFTO : number of cases that no left-turning car left behind in the system by an arbitrary test car

NX, NY, NZ : number of times that X, Y, and Z are generated

TX, TY, TZ : total time units for X, Y, and Z in this simulation

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APPENDIX A

TWT : total (accumulated) waiting time experienced by the whole left-turning cars appeared in this simulation

IND : dummy variable (index) to generate a random number

XX : arrival rate (cars/sec) of successive cars

X : time of next arrival of successive cars (sec/car)

YY\textsuperscript{-1} : crossing (street) rate (cars/sec) of left-turning cars given that no blocking is seen by the successive cars

Y : crossing (street) time of a left-turning car

ZZ : arrival rate of left-turning cars (cars/sec)

Z : time of next arrival of left-turning cars (sec/car)

N : number of trials in this simulation

W : mean service time of a left-turning car from the instant that he faces traffic till he completely crosses street

V : theoretical variance of service time of a left-turning car from the instant he faces traffic till he completely crosses street

PBLK : the probability of blocking
Waiting Time Distribution and Entropy (Average Uncertainty) on Left-Turning against Traffic [Part II] (Tokuichi Yonemori)

APPENDIX B

SOURCE PROGRAM

MIDIM Model

--- THIS IS THE MIDIM MODEL ---

This simulation obtains the mean and variance of number of left-turning cars left behind by a left-turning test car in the MIDIM model (Poisson (Markov) arrivals of successive cars. Constant (deterministic) crossing time of a left-turning car. Poisson arrivals of left-turning cars).

XX determines X, time of next arrival of successive cars
YY determines Y, crossing time of left-turning cars
ZZ determines Z, time of next arrival of left-turning car

XX, YY, and ZZ could have some values between 5 cars/min (0.1 car/sec) and 55 cars/min (0.9 cars/sec)

Dimension Left(100)

--- SET INITIAL CONDITIONS ---

For detail discussion of variables, see Appendix A

Data Left, NTRIAL, NQUE, LEFT0, NX, NY, NZ, IND/100*0.7*0/
Data QUE, QUE2, TX, TY, TZ, TWT/6*0.0/

--- GIVE ARBITRARY VALUES FOR XX, YY, ZZ, AND N ------

READ(1,73)XX, YY, ZZ, N
73 FORMAT(3F3.1,13)
WRITE(2,74)XX, YY, ZZ, N
74 FORMAT(2X,3HXX=.F3.1,2X.3HYY=,11X.17HNUMBER OF TRIALS=,I5/)

--- GENERATE X, Y, AND Z ----

98 CALL SUB2(IND, X, XX, TX, NX, 1)
CALL SUB2(IND, Y, YY, TY, NY, 2)
CALL SUB2(IND, Z, ZZ, TZ, NZ, 1)

--- CHECK WHAT HAPPENS TO THE LEFT-TURNING CAR ----

99 CONTINUE
IF(Z.GE.X AND X.GT.Y) GOTO 1
IF(X.GT.Z AND Z.GT.Y) GOTO 1
IF(X.GT.Y AND Y.GT.Z) GOTO 6
IF(Z.GT.Y AND Y.GE.X) GOTO 4
IF(Y.GE.Z AND Z.GT.X) GOTO 4
IF(Y.GE.X AND X.GT.Z) GOTO 7
IF(Y.GE.X AND X.EQ.Z) GOTO 8
IF(X.GT.Y AND Y.EQ.Z) GOTO 9
WRITE(2,111)
111 FORMAT(5X,5HERROR/)
GOTO 1110
C-----THE LEFT-TURNING TEST CAR FINALLY PASSES STREET-----

C
1 IF(NQUE) 101, 101, 102
101 CALL SUB1(NQUE, LEFT0, LEFT, NTRIAL, QUE, QUE2)
   TWT=TWT+Y
   G0 TO 97
102 CALL SUB2(NQUE, LEFT0, LEFT, NTRIAL, QUE, QUE2)
   TWT=TWT+Y
   G0 TO 97

C-----WHILE HE IS PASSING STREET, ANOTHER LEFT-TURNING
C CAR JOIN THE QUEUE-------

C
6 TWT=TWT+Z
   NQUE=NQUE+1
   X=X-Z
   Y=Y-Z
   CALL SUB2(IND, Z, ZZ, TZ, NZ, 1)
   G0 TO 99

C-----SINCE HE CANNOT PASS, HE WAITS FOR ANOTHER TRIAL------

C
4 TWT=TWT+X
   CALL SUB2(IND, X, XX, TX, NX, 1)
   CALL SUB2(IND, Y, YY, TY, NY, 2)
   CALL SUB2(IND, Z, ZZ, TZ, NZ, 1)
   G0 TO 99

C-----WHILE HE IS WAITING FOR ANOTHER TRIAL, ANOTHER LEFT-
C TURNING CAR JOIN THE QUEUE-------

C
7 TWT=TWT+Z
   X=X-Z
   NQUE=NQUE+1
   Y=Y-Z
   CALL SUB2(IND, Z, ZZ, TZ, NZ, 1)
   G0 TO 99

C-----WHEN HE BEGINS NEXT TRIAL, HE SEES ANOTHER LEFT-
C TURNING CAR JOIN THE QUEUE-------

C
8 TWT=TWT+X
   NQUE=NQUE+1
   CALL SUB2(IND, X, XX, TX, NX, 1)
   CALL SUB2(IND, Y, YY, TY, NY, 2)
   CALL SUB2(IND, Z, ZZ, TZ, NZ, 1)
   G0 TO 99

C
9 NQUE=NQUE+1
   G0 TO 102

C-----CHECK IF THE NUMBER OF TRIALS IS EQUAL TO N---------

C
97 IF(NTRIAL-N) 98, 77, 77
Midim Model

C----Now we get the exact values of xx, yy, and zz in this simulation -----

77 Continul
  xx=flcat(nx)/tx
  yy=flcat(ny)/ty
  zz=flcat(nz)/tz

C  Write(2.49)xx,yy,zz
  49 Format(1x,13htheoretically/
         11x,3hx=,f6.3,3x,3hy=,f6.3,3x,3hz=,f6.3/)

C----For the theoretical results, we use previously obtained
C values for y,w,pblk,v, and eq (see the paper (1977) by
C yonemori ----

C  y = 1.0/yy
  w=(exp(xx*y)-1.0)/xx
  pblk=1.0-exp(-xx*y)
  row=zz*w
  if(row.lt.1.0) go to 1010
  write(2.20)
  20 Format(2x,17hOver flow is seen/)

C  1010 continue
  v=(exp(2.0*xx*y)-2.0*xx*y*exp(xx*y)-1.0)/xx**2
  eq=row**2*(1.0+v/w**2)/(2.0*(1.0-row))

C  Write(2.1091)y,w,row,eq,pblk
  1091 Format(2x,2hy=,e15.7,5x,2hw=,e15.7,5x,4hrw=,e15.7/
           12x,3heq=,e15.7,5x,5hpblk=,e15.7/)

C----These are the mathematical equations for variance of
C number of left-turning cars left behind by a test car ----

C  heikin= row + eq
  h1=zz*(exp(xx*y)-1.0)/xx
  h2=2.0*zz/xx)**2*exp(xx*y)*(exp(xx*y)-xx*y-1.0)
  h3=2.0*xx*y*(xx*y)**2-2.0*exp(xx*y)
  h4=2.0*exp(2.0*xx*y)-4.0*xx*y*exp(xx*y)
  h5=3.0*zz/xx)**3*exp(xx*y)**h
  g1=h2/(1.0-h)+h2**2/(2.0*(1.0-h)**2)+h3/(3.0*(1.0-h))
  vary=g1+heikin-heikin**2

C  Write(2.1095)heikin,vary
  1095 Format(2x,7hheikin=,e15.7,5x,5hvary=,e15.7/)

C----We get the simulation results for the mean and variance of
C the number of left-turning cars left behind by a test car ---

APPENDIX B
(PAGE 4)

MIDIM MODEL

T=QUE/FLOAT(NTRIAL)
TV=QUE2/FLOAT(NTRIAL)-T**2
AWT=TWT/FLOAT(NTRIAL)
WRITE(2,50)T,TV,AWT
50 FORMAT(1X,13HBY SIMULATION/
11X*,7HEIKIN=.*E15.7,5X,*5HVARY=.*E15.7,5X,*3HAWT=.*E15.7/)

C----IN THIS SECTION. WE GET THE PROBABILITIES PO,P1,P2.....
C FROM THIS SIMULATION -----

PO=FLOAT(LEFTO)/FLOAT(NTRIAL)
WRITE(2,5)LEFTO,PO
5 F0RMAT(1X,4H 0.I4,F9.3)
DO 999 1=1,20
P=FLOAT(LEFT(I))/FLOAT(NTRIAL)
WRITE(2,55)I,LEFT(I),P
55 FORMAT(1X,2I4,F9.3)
999 CONTINUE

C
1094 WRITE(2,19)
19 F0RMAT(3X,5HVAR1/)
1110 STOP

C

SUBROUTINE SUB(IND,RAND)
C THIS SUBROUTINE GENERATES A RANDOM NUMBER------
C THIS PROGRAM IS QUOTED FROM PAGE 197 "FORTRAN
C PROGRAMMING" (1972) BY K. MATSUMOTO.
C
IF(IND-D100.100.200
100 XX=5024934.
AK = 23.
AM=1.0E7+1.

C
200 W = XX * AK
RN= AMOD(W,AM)
XX=RN
RANDE=RN/1.0E7
C
RETURN
END

C

SUBROUTINE SUBK(N,QUE,LEFTO,LEFT,NTRIAL,QUE,QUE2)
C----THIS SUBROUTINE MAINLY ACCUMULATES THE VALUES FOR PAIO.
C PAI1,PAI2..... FOR THIS SIMULATION-----
C
DIMENSION LEFT(100)

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C GO TO (1.2).N
C
C FOR NO CAR LEFT BEHIND ----
C 1 LEFTO=LEFTO + 1
   NTRIAL = NTRIAL + 1
   NQUE=0
   GO TO 7
C
C FOR MORE THAN ONE CAR LEFT BEHIND ----
C 2 LEFT(NQUE) = LEFT(NQUE) + 1
   QUE=QUE+FLOAT(NQUE)
   QUE2=QUE2+ FLOAT(NQUE**2)
   NTRIAL = NTRIAL + 1
   NQUE=NQUE-1
C
7 RETURN
END
C
C SUBROUTINE SUB2(IND,W,WW,TW,NW,N)
C THIS SUBROUTINE DETERMINES X,Y,AND Z----
C GO TO (1.2).N
C
C FOR POISSON ARRIVAL AND POISSON SERVICE TIME ----
C 1 IND=IND+1
   CALL SUB(IND, RAND)
   W = (-1.0/WW)*ALOG(RAND)
   GO TO 3
C
C FOR CONSTANT ARRIVAL AND CONSTANT SERVICE TIME ----
C 2 W=1.0/WW
C
C FOR CHECKING EXACT VALUES OF XX,YY,AND ZZ ----
C 3 TW=TW+W
   NW=NW+1
C
RETURN
END
SOURCE PROGRAM

MIMIM MODEL

------------------ THIS IS THE MIMIM MODEL ------------------

THIS SIMULATION OBTAINS THE MEAN AND VARIANCE OF NUMBER
OF LEFT-TURNING CARS LEFT BEHIND BY A LEFT-TURNING TEST
CAR IN THE MIMIM MODEL (POISSON (MARKOV) ARRIVALS OF
SUCCESSIVE CARS. POISSON (MARKOV) CROSSING TIME OF A
LEFT-TURNING CAR. POISSON (MARKOV) ARRIVALS OF LEFT-
TURNING CARS ).

XX DETERMINES X, TIME OF NEXT ARRIVAL OF SUCCESSIVE CARS
YY DETERMINES Y, CROSSING TIME OF LEFT-TURNING CARS
ZZ DETERMINES Z, TIME OF NEXT ARRIVAL OF LEFT-TURNING CAR

XX, YY, AND ZZ COULD HAVE SOME VALUES BETWEEN 5 CARS/MIN
(0.1 CAR/SEC) AND 55 CARS/MIN (0.9 CAR/SEC)

DIMENSION LEFT(100)

---SET INITIAL CONDITIONS---
FOR DETAIL DISCUSSION OF VARIABLES, SEE APPENDIX A

DATA LEFT, NTRIAL, NQUE, LEFTO, NX, NY, NZ, IND/100*0.7*0/
DATA QUE, QUE2, TX, TY, TZ, TWT/6*0.0/

---GIVE ARBITRARY VALUES FOR XX, YY, ZZ, AND N ---

READ(1,73)XX, YY, ZZ, N
73 FORMAT(3F3.1,13)
WRITE(2,74)XX, YY, ZZ, N
74 FORMAT(2X,3HXX=.F3.1,2X.3HHYY=.F3.1,2X.3HZZ=.F3.1/
11X,17HNUMBER OF TRIALS=.I5/)

---GENERATE X, Y, AND Z ----

98 CALL SUB2(IND, X, XX, TX, NX, 1)
CALL SUB2(IND, Y, YY, TY, NY, 1)
CALL SUB2(IND, Z, ZZ, TZ, NZ, 1)

---CHECK WHAT HAPPENS TO THE LEFT-TURNING CAR----

99 CONTINUE
IF(Z.GE.X.AND.X.GT.Y) GO TO 1
IF(X.GT.Z.AND.Z.GT.Y) GO TO 1
IF(X.GT.Y.AND.Y.GT.Z) GO TO 6
IF(Z.GT.Y.AND.Y.GE.X) GO TO 4
IF(Y.GE.Z.AND.Z.GT.X) GO TO 4
IF(Y.GE.X.AND.X.GT.Z) GO TO 7
IF(Y.GE.X.AND.X.EQ.Z) GO TO 8
IF(X.GT.Y.AND.Y.EQ.Z) GO TO 9
WRITE(2,111)
111 FORMAT(5X,5FERROR/)
GO TO 1110
Waiting Time Distribution and Entropy (Average Uncertainty) on Left-Turning against Traffic [Part II] (Tokuichi Yonemori)

APPENDIX C
(PAGE 2)

MIMIM MODEL

C C---- THE LEFT-TURNING TEST CAR FINALLY PASSES STREET -----
C
1 IF(NQUE) 101.101.102
  101 CALL SUB1(1,NQUE,LEFTO,LEFT,NTRIAL,QUE,QUE2)  
      TWT=iwT+Y  
      GO TO 97
  102 CALL SUB1(2,NQUE,LEFTO,LEFT,NTRIAL,QUE,QUE2)  
      TWT=TWT+Y  
      GO TO 97
C C---- WHILE HE IS PASSING STREET, ANOTHER LEFT-TURNING C  
C CAR JOIN THE QUEUE ------
C
6 TWT=TWT+Z  
   NQUE=NQUE+1  
   X=X-Z  
   Y=Y-Z  
   CALL SUB2(IND,Z,ZZ,TZ,NZ,1)  
   GO TO 99
C C---- SINCE HE CANNOT PASS, HE WAITS FOR ANOTHER TRIAL ------
C
4 TWT=TWT+X  
   CALL SUB2(IND,X,XX,TX,NX,1)  
   CALL SUB2(IND,Y,YY,TY,NY,1)  
   CALL SUB2(IND,Z,ZZ,TZ,NZ,1)  
   GO TO 99
C C---- WHILE HE IS WAITING FOR ANOTHER TRIAL, ANOTHER LEFT-  
C TURNING CAR JOIN THE QUEUE ------
C
7 TWT=TWT+Z  
   X=X-Z  
   NQUE=NQUE+1  
   Y=Y-Z  
   CALL SUB2(IND,Z,ZZ,TZ,NZ,1)  
   GO TO 99
C C---- WHEN HE BEGINS NEXT TRIAL, HE SEES ANOTHER LEFT-  
C TURNING CAR JOIN THE QUEUE ------
C
8 TWT=TWT+X  
   NQUE=NQUE+1  
   CALL SUB2(IND,X,XX,TX,NX,1)  
   CALL SUB2(IND,Y,YY,TY,NY,1)  
   CALL SUB2(IND,Z,ZZ,TZ,NZ,1)  
   GO TO 99
C
9 NQUE=NQUE+1  
   GO TO 102
C C---- CHECK IF THE NUMBER OF TRIALS IS EQUAL TO N -------
C
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97 IF(NTRIAL-N) 98,77,77
C
C----NOW WE GET THE EXACT VALUES OF XX, YY, AND ZZ
C IN THIS SIMULATION ------
C
77 CONTINUE
XX=FLOAT(NX)/TX
YY=FLOAT(NY)/TY
ZZ=FLOAT(NZ)/TZ
C
WRITE(2,49)XX,YY,ZZ
49 FORMAT(IX,13HTHEORETICALLY/
11X,3HXX=.,F6.3.3X.3HYY=.,F6.3X.3HZZ=.,F6.3/
C
C----FOR THE THEORETICAL RESULTS, WE USE PREVIOUSLY OBTAINED
C VALUES FOR W, PBLK, V, AND EQ (SEE THE PAPER (1977) BY
C YONEMORI -----
C
PBLK=XX/(XX+YY)
W=1.0/YY
ROW=ZZ*W
IF(ROW.LT.1.0) GO TO 1010
WRITE(2,20)
20 FORMAT(3X,17H0VER FLOW IS SEEN/)
C
1010 CONTINUE
EQ=ROW**2/(1.0-ROW)
HEIKIN=ROW+EQ
VARY=ROW/(1.0-ROW)**2
C
WRITE(2,1091)W,ROW,EQ,PBLK,HEIKIN,VARY
1091 FORMAT(2X,1H W=.E15.7.5X.4HR0W=.E15.7.5X.3HEQ=.E15.7.5X/
12X,5HPBLK=.E15.7.5X.7HEIKIN=.E15.7.6X.5HVARY=.E15.7/)
C
C----WE GET THE SIMULATION RESULTS FOR THE MEAN AND VARIANCE OF
C THE NUMBER OF LEFT-TURNING CARS LEFT BEHIND BY A TEST CAR ---
C
T=QUE/FLOAT(NTRIAL)
TV=QUE2/FLOAT(NTRIAL)-T**2
AWT=TWT/FLOAT(NTRIAL)
WRITE(2,50)T,TV,AWT
50 FORMAT(IX,13HZBY SIMULATION/
11X,7HEIKIN=.E15.7.5X.5HVARY=.E15.7.5X.3HAWT,.E15.7/)
C
C----IN THIS SECTION, WE COMPARE THE PROBABILITIES PAI0, PAI1,
C PAI2, ---- OBTAINED FROM MATHEMATICAL ANALYSIS WITH
C P0, P1, P2, ---- OBTAINED FROM THIS SIMULATION ------
C
P0=FLOAT(LEFT0)/FLOAT(NTRIAL)
PAI0=1.0-ROW

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APPENDIX C

(MIMIM MODEL)

WRITE(2,5)LEFTO,P0,PAI0
5 FORMAT(1X,4H 0,14,2F9.3)
DO 999 I=1,20
P=FLOAT(LEFT(I))/FLOAT(NTRIAL)
PAI =PAI0*ROW**I
WRITE(2,55)I,LEFT(I),P,PAI
55 FORMAT(1X,2I4,2F9.3)
999 CONTINUE
C
1094 WRITE(2,19)
19 FORMAT(2X,5H0WARI/)
1110 STOP
END
C
C
SUBROUTINE SUB(IND,RAND)
C
C THIS SUBROUTINE GENERATES A RANDOM NUMBER------
C (THIS PROGRAM IS QUOTED FROM PAGE 197 "FORTRAN
C PROGRAMMING" (1972) BY KINJI MATSUMOTO)
C
100 XX=5024934.
   AK = 23.
   AM=1.0E7+1.
200 W = XX * AK
   RN= AMOD(W,AM)
   XX=RN
   RAND=RN/1.0E7
RETURN
END
C
C
SUBROUTINE SUB1(N,NQUE,LEFTO,LEFT,NTRIAL,QUE,QUE2)
C
C THIS SUBROUTINE MAINLY ACCUMULATES THE VALUES FOR PAI0,
C PAI1,PAI2------ FOR THIS SIMULATION------
C
DIMENSION LEFT(100)
G0 TO (1.2),N
C
C FOR NO CAR LEFT BEHIND ----
C
1 LEFTO=LEFTO + 1
NTRIAL = NTRIAL + 1
NQUE=0
G0 TO 7
MIMIM Model

--- FOR MORE THAN ONE CAR LEFT BEHIND ---

\[
\begin{align*}
2 \text{ LEFT}(\text{NQUE}) &= \text{LEFT}(\text{NQUE}) + 1 \\
\text{QUE} &= \text{QUEUE} + \text{FLOAT}(\text{NQUE}) \\
\text{QUE}^2 &= \text{QUEUE}^2 + \text{FLOAT}(\text{NQUE}^2) \\
\text{NTRIAL} &= \text{NTRIAL} + 1 \\
\text{NQUE} &= \text{NQUE} - 1
\end{align*}
\]

7 RETURN
END

SUBROUTINE SUB2(IND, WW, TW, NW, N)

--- THIS SUBROUTINE DETERMINES X, Y, AND Z ---

GO TO (1, 2) * N

--- FOR POISSON ARRIVAL AND POISSON SERVICE TIME ---

1 IND = IND + 1
CALL SUB2(IND, RAND)
W = (-1.0 / WW) * ALOG(RAND)
GO TO 3

--- FOR CONSTANT ARRIVAL AND CONSTANT SERVICE TIME ---

2 W = 1.0 / WW

--- FOR CHECKING EXACT VALUES OF XX, YY, AND ZZ ---

3 TW = TW + W
NW = NW + 1
RETURN
END