Title	The algebraic and geometric classification of generically non-degenerate projections of matrix algebras over complex numbers
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Citation	琉球大学理学部紀要 = BULLETIN OF THE FACULTY OF SCIENCE UNIVERSITY OF THE RYUKYUS(107): 1-31
Issue Date	2019-03-29
URL	http://hdl.handle.net/20.500.12000/44488
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The algebraic and geometric classification of generically non-degenerate projections of matrix algebras over complex numbers

Takahiro Sudo

Abstract

We study projections of matrix algebras over complex or real numbers, to (partially) classify the (generically non-degenerate) projections (in our sense) and their spaces algebraically and geometrically, in the three by three, and four by four matrix cases, and in the general matrix case.

Primary 15B57, 46L05, 46L80, 19K14,

Keywords: Matrix algebra, Bott projection, K-theory, noncommutative topology, projection, trace, algebraic equation, complex number.

1 Introduction

We study projections of matrix algebras over complex or real numbers, to (partially) classify the (generically non-degenerate) projections (in our sense) and their spaces algebraically and geometrically, in the three by three, and four by four matrix cases, and in the general matrix case.

As obtained in [3], as the first step (in our sense), we have already studied projections of matrix algebras over complex or real numbers, to classify the generalized Bott projections (in our sense) and their spaces algebraically and geometrically, in the same cases as above, involving and generalizing the so called Bott projection (cf. [6]), which plays a crucial role in the K-theory of C^*-algebras (cf. [4] and [6]).

In this paper, as the second step (unexpected at en early stage), we obtain algebraic and geometric classification (partial) results (by examples) on the (generically non-degenerate) projections (in our sense) of matrix algebras over complex or real numbers and their spaces in the 3×3, and 4×4 matrix cases, and in the general matrix case. For this, we review the 2×2 matrix case, involving the Bott projection (cf. [6]). May as well refer to [4, 8.5], [5], and [1]. There may be other items found in the literature, but such would be not so many.

Received January 21, 2019.

The explicit formulae obtained by determining those (non-degenerate) projections algebraically may be some useful as a convenient reference. As well, the geometric (or topological) structure for the spaces of the (non-degenerate) projections may have some applications such as to the theory of C^*-algebras. Certainly, the geometric (or topological) structure for the spaces may be considered as the second basic step towards yet a noncommutative geometry (or topology) theory for C^*-algebras (without stabilizing), such as the (stabilized) K-theory for C^*-algebras.

Moreover, more further computation may be involved and needed to improve and refine the results obtained so far as in this paper, towards the goal as the more general classification results. This task may (or not) be continued in the future.

An early version (such as [3]) with the title slightly different from that of this paper has been reviewed, from which this paper is somewhat improved, extended, and separated from the version (as [3] also separated).

Notation. We use the symbol $\equiv$ as a definition. We use the symbol $\approx$ as a homemorphism. Denote by $M_n(\mathbb{C})$ the $n \times n$ matrix Banach or C^*-algebra over $\mathbb{C}$ of complex numbers. For convenience, we may use the Euclidean norm on $M_n(\mathbb{C})$ as the complex n^2-dimensional Euclidean vector space $\mathbb{C}^{n^2}$, to equip its topolology. Let $i \in \mathbb{C}$ with $i^2 = -1$. Denote by $M_n(\mathbb{R})$ the $n \times n$ matrix Banach algebra over $\mathbb{R}$ of real numbers.

Recall that an element $p \in M_n(\mathbb{C})$ is a projection if and only if $p = p^2 = p^*$ with p^* the adjoint of p, i.e, the complex conjugate transpose of p. We denote by $P(M_n(\mathbb{C}))$ the space of all non-trivial projections of $M_n(\mathbb{C})$ with relative topology, and by

$$P(M_n(\mathbb{C}))^\sim = P(M_n(\mathbb{C})) \sqcup \{0_n, 1_n\} \quad \text{(as a disjoint union)}$$

the space of all projections of $M_n(\mathbb{C})$, where 0_n is the $n \times n$ zero matrix and 1_n is the $n \times n$ identity matrix. Define $P(M_n(\mathbb{R}))$ and $P(M_n(\mathbb{R}))^\sim$ similarly.

For $a, b \in M_n(\mathbb{C})$ ($n \geq 1$), we denote by $a \oplus b$ the diagonal sum in $M_{2n}(\mathbb{C})$.

For any $a = (a_{ij}) \in M_n(\mathbb{C})$, denote by $\mathrm{tr}(a) = \sum_{j=1}^n a_{jj}$ the canonical trace of a.

It follows from the Linear Algebra theory that for any self-adjoint (or Hermitian) matrix $a \in M_n(\mathbb{C})$ with $a = a^*$, there is a unitary matrix u in $M_n(\mathbb{C})$ such that uau^* is diagonal with real entries on the diagonal (cf. [2]). In particular, if p is a projection of $M_n(\mathbb{C})$, there is a unitary u such that $upu^* = 1_k \oplus 0_{n-k}$ for some $0 \leq k \leq n$, with 1_0 and 0_0 removed, so that

$$\mathrm{tr}(p) = \mathrm{tr}(pu^*u) = \mathrm{tr}(upu^*) = k.$$

2 The 2×2 matrix case, reviewed

By solving the equation for the definition of projections of $M_2(\mathbb{C})$, we obtain that, with several proper notations according to the calculation in the proof below,

Theorem 2.1. ([3]). *If $p = (p_{ij})_{i,j=1}^2$ is a non-trivial projection of $M_2(\mathbb{C})$ with $0_2 \neq p = p^2 = p^* \neq 1_2$, then p is either*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 \oplus 0 \equiv p_1, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0 \oplus 1 \equiv p_2, \quad or$$

$$\begin{pmatrix} a = a^{\pm}(z) \equiv \frac{1 \pm \sqrt{1-4|z|^2}}{2} & \overline{z} \\ z & b = b^{\mp}(z) \equiv \frac{1 \mp \sqrt{1-4|z|^2}}{2} \end{pmatrix} \equiv p_{\pm}(z)$$

(in compound order), for any $z \in \mathbb{C} \setminus \{0\}$, with $0 \leq 1 - 4|z|^2 < 1$ if and only if $0 < |z| \leq \frac{1}{2}$. We may as well define $p_{\pm}(0)$ as

$$p_+(0) = \lim_{z \to 0} p_+(z) = p_1 \quad and \quad p_-(0) = \lim_{z \to 0} p_-(z) = p_2.$$

For $z = \frac{1}{2}e^{i\theta} \in \mathbb{C}$ with $\theta \in \mathbb{R}$ (mod 2π) and $|z| = \frac{1}{2}$,

$$p_{\pm}(\frac{1}{2}e^{i\theta}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}e^{-i\theta} \\ \frac{1}{2}e^{i\theta} & \frac{1}{2} \end{pmatrix} \equiv p(\frac{1}{2}e^{i\theta}).$$

In particular,

$$p_+(\pm\frac{1}{2}) = p_-(\pm\frac{1}{2}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \equiv p(\frac{1}{2}) \in M_2(\mathbb{R}).$$

Moreover, $p_+(z) = p_-(z)$ for $z \in \mathbb{C} \setminus \{0\}$ if and only if $|z| = \frac{1}{2}$. Furthermore, $p_{\pm}(z) \in M_2(\mathbb{R})$ if and only if $z \in \mathbb{R}$. Also, the extended $p_{\pm}(z)$ for $0 \leq |z| \leq \frac{1}{2}$ are viewed as (respectively) injective, continuous, projection-valued functions from the real 2-dimensional closed ball $\{z \in \mathbb{C} \,|\, |z| \leq \frac{1}{2}\} \equiv B^2(\frac{1}{2})$ in $\mathbb{C}$ with center zero and radius $\frac{1}{2}$ to $P(M_2(\mathbb{C}))$.

For the proof, may refer to the next section.

Corollary 2.2. ([3]). *The space $P(M_2(\mathbb{R}))$ consists of p_1, p_2, and $p_{\pm}(t)$ for $t \in \mathbb{R}$ with $0 < |t| \leq \frac{1}{2}$, where $p_1 = \lim_{t \to 0} p_+(t) \equiv p_+(0)$, $p_2 = \lim_{t \to 0} p_-(t) = p_-(0)$, and $p_+(\pm\frac{1}{2}) = p(\frac{1}{2}) = p_-(\pm\frac{1}{2})$.*

Now let X and Y be topological spaces and K be a space viewed as a subspace in both X and Y. We denote by $X \sqcup_K Y$ the **K-jointed sum** (or the connected sum) of X and Y (on K) which is defined to be the space obtained from attaching X and Y on the space K, or in other words, as that the space K viewed in X is identified with K viewed in Y, in the disjoint union $X \sqcup Y$.

We denote by S^2 the real 2-dimensional sphere in $\mathbb{R}^3$.

Theorem 2.3. ([3]). *There is a homeomorphism between the space $P(M_2(\mathbb{C}))^{\sim}$ and the disjoint union $\{0_2\} \sqcup \{1_2\} \sqcup (B^2(\frac{1}{2}) \sqcup_{S^1(\frac{1}{2})} B^2(\frac{1}{2}))$, with*

$$S^2 \approx B^2(\frac{1}{2}) \sqcup_{S^1(\frac{1}{2})} B^2(\frac{1}{2}) \approx P(M_2(\mathbb{C}))$$

of all rank 1 *projections of* $M_2(\mathbb{C})$*, where* $B^2(2^{-1}) \sqcup_{S^1(2^{-1})} B^2(2^{-1})$ *is the space obtained from attaching two copies of* $B^2(2^{-1})$ *along the set* $S^1(\frac{1}{2}) = \{z \in \mathbb{C} \,|\, |z| = \frac{1}{2}\}$*, as a* $S^1(2^{-1})$*-jointed sum or a circle-jointed sum, where* $S^1(2^{-1})$ *is homeomorphic to the real* 1*-dimensional sphere* $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$*.*

Proof. Indeed, the homeomorphism is given as

$$z \sqcup_{S^1(\frac{1}{2})} w \mapsto p_+(z) \sqcup_{p_\pm(S^1(\frac{1}{2}))} p_-(w),$$

for $z, w \in \mathbb{C}$, with $0 \le |z| \le \frac{1}{2}$ and $0 \le |w| \le \frac{1}{2}$, where z and w are identified on $S^1(2^{-1})$, and the image has the similar meaning. □

Corollary 2.4. ([3]). *There is a homeomorphism between the space* $P(M_2(\mathbb{R}))^\sim$ *and the disjoint union* $\{0_2\} \sqcup \{1_2\} \sqcup ([-\frac{1}{2}, \frac{1}{2}] \sqcup_{S^0(\frac{1}{2})} [-\frac{1}{2}, \frac{1}{2}])$*, with*

$$S^1 \approx ([-\frac{1}{2}, \frac{1}{2}] \sqcup_{S^0(\frac{1}{2})} [-\frac{1}{2}, \frac{1}{2}]) \approx P(M_2(\mathbb{R}))$$

of all rank 1 *projections of* $M_2(\mathbb{R})$*, where* $[-2^{-1}, 2^{-1}] \sqcup_{S^0(2^{-1})} [-2^{-1}, 2^{-1}]$ *is the space obtained from attaching two copies of the closed interval* $[-2^{-1}, 2^{-1}]$ *at the set* $S^0(\frac{1}{2}) = \{t \in \mathbb{R} \,|\, |t| = \frac{1}{2}\} = \{\pm\frac{1}{2}\}$*, as a two-points-jointed sum, where* $S^0(2^{-1})$ *is homeomorphic to the real* 0*-dimensional sphere* $S^0 = \{t \in \mathbb{R} \,|\, |t| = 1\} = \{\pm 1\}$*.*

Table 1: The involved signs, only points, or limits at zeros of the diagonal components $a = a^\pm(z)$ and $b = b^\pm(z)$ with the off diagonal conditions of z

Trace	Sign	Diagonal $(a^\pm, b^\pm)$	Off diagonal $(z, \overline{z})$
0	$(-,-)$	Point $(0,0)$	$z = 0$
1	$(+,-)$	Point $(1,0)$ as	$z = 0$
	$(+,-)$	Limit $\lim_{z\to 0, z\neq 0}(a^+(z), b^-(z))$	$z \neq 0$
	$(-,+)$	Point $(0,1)$ as	$z = 0$
	$(-,+)$	Limit $\lim_{z\to 0, z\neq 0}(a^-(z), b^+(z))$	$z \neq 0$
2	$(+,+)$	Point $(1,1)$	$z = 0$

3 The 3×3 matrix case

Solving the equation for the definition of projections implies that

Lemma 3.1. *If* $p = (p_{ij})$ *is a projection of* $M_3(\mathbb{C})$*, then*

$$p = p(z_1, z_2, z_3) \equiv \begin{pmatrix} a & \overline{z_1} & \overline{z_2} \\ z_1 & b & \overline{z_3} \\ z_2 & z_3 & c \end{pmatrix} \quad a, b, c \in \mathbb{R}, z_1, z_2, z_3 \in \mathbb{C},$$

where, as the condition (D_3) *on the diagonal part,*

$$a^2 + |z_1|^2 + |z_2|^2 = a, \quad |z_1|^2 + b^2 + |z_3|^2 = b, \quad |z_2|^2 + |z_3|^2 + c^2 = c,$$

and, as the condition (O_3) *on the off diagonal part,*

$$(a+b)\overline{z_1} + \overline{z_2}z_3 = \overline{z_1}, \quad (a+c)\overline{z_2} + \overline{z_1 z_3} = \overline{z_2}, \quad z_1\overline{z_2} + (b+c)\overline{z_3} = \overline{z_3},$$

so that the condition (D_3) *implies that, as the solution condition* (S_3)*,*

$$\begin{aligned} a = a^{\pm}(z_1, z_2) &\equiv \frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_1|^2 - |z_2|^2} \quad \text{if } 0 \le |z_1|^2 + |z_2|^2 \le \frac{1}{4}, \\ b = b^{\pm}(z_1, z_3) &\equiv \frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_1|^2 - |z_3|^2} \quad \text{if } 0 \le |z_1|^2 + |z_3|^2 \le \frac{1}{4}, \\ c = c^{\pm}(z_2, z_3) &\equiv \frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_2|^2 - |z_3|^2} \quad \text{if } 0 \le |z_2|^2 + |z_3|^2 \le \frac{1}{4}. \end{aligned}$$

Moreover, as the equality condition (E_3), $a^+(z_1, z_2) = a^-(z_1, z_2)$ *if and only if* $|z_1|^2 + |z_2|^2 = \frac{1}{4}$, *and* $b^+(z_1, z_3) = b^-(z_1, z_3)$ *if and only if* $|z_1|^2 + |z_3|^2 = \frac{1}{4}$, *and* $c^+(z_2, z_3) = c^-(z_2, z_3)$ *if and only if* $|z_2|^2 + |z_3|^2 = \frac{1}{4}$.

Corollary 3.2. *It follows from Lemma* 3.1 *the following*:

(1) *If* $z_1 = z_2 = z_3 = 0$, *then* $a = 0$ *or* 1, *and* $b = 0$ *or* 1, *and* $c = 0$ *or* 1.

(2) *If* $z_1 \ne 0$ *and* $z_2 = z_3 = 0$, *or if* $z_1 = z_2 = 0$ *and* $z_3 \ne 0$, *then these correspond to the* 2×2 *case. If* $z_1 = z_3 = 0$ *and* $z_2 \ne 0$, *then this also correspond to the* 2×2 *case similarly.*

(1) *or* (2) *If* $z_1 = 0$ *is zero, then* z_2 *or* z_3 *is zero. If* $z_2 = 0$ *is zero, then* z_1 *or* z_3 *is zero. If* $z_3 = 0$, *then* z_1 *or* z_2 *is zero.*

(3) *The rest is the case where* z_1, z_2, z_3 *are all non-zero.*

Proof. Use the conditions (S_3) and (O_3). □

We may say that a projection $p \in M_3(\mathbb{C})$ is **full degenerate** (on the off diagonal part) if it is in the case (1), and is either **partially degenerate** or a **generalized Bott** projection if in the case (2), and is **generically non-degenerate** (on the off diagonal part) if in the case (3).

Lemma 3.3. *In Lemma* 3.1, *if* z_1 *and* z_2 *in* $\mathbb{C}$ *are fixed with* $0 \le |z_1|^2 + |z_2|^2 \le \frac{1}{4}$, *then*

$$0 \le |z_3|^2 \le \min\{\frac{1}{4} - |z_1|^2, \frac{1}{4} - |z_2|^2\} \quad \text{as the minimum.}$$

This condition, as the bounded condition (B_3)*, also holds by exchanging or permuting coordinates* z_1, z_2*, and* z_3.

We denote by $B^k(r)$ the real k-dimensional closed ball with radius $r > 0$ center zero, by $S^k(r)$ the real k-dimensional sphere with radius $r > 0$, and set $S^k(0) = \{0_{k+1}\}$ with $0_{k+1} \in \mathbb{R}^{k+1}$ the zero.

Proposition 3.4. *Each connected component of the space $P(M_3(\mathbb{C}))^\sim$ is contained in the path-connected bounded set $S^{4,2}$ defined by the diagonal condition (D_3) as the bounded condition (B_3) above, which is homeomorphic to a fiber space over the base space $S^4(\frac{1}{2})$ identified with $B^4(\frac{1}{2}) \sqcup_{S^3(\frac{1}{2})} B^4(\frac{1}{2})$, with fibers $S^2(\sqrt{4^{-1} - r^2})$ identified with*

$$B^2(\sqrt{4^{-1} - r^2}) \sqcup_{S^1(\sqrt{4^{-1}-r^2})} B^2(\sqrt{4^{-1} - r^2})$$

for $0 \leq r \leq \frac{1}{2}$, where r is in fact defined as $r = \max\{|z_1|, |z_2|\}$ with $0 \leq |z_1|^2 + |z_2|^2 \leq 4^{-1}$. Namely,

$$\begin{array}{ccc} S^{4,2} & \longleftarrow & S^2(\sqrt{4^{-1} - r^2}) \\ \big\downarrow & & \\ S^4(\frac{1}{2}) & & \end{array}$$

with $S^4(\frac{1}{2})$ the base space and $S^2(\sqrt{4^{-1} - r^2})$ as fibers, which looks like a space obtained by attaching two sorts of cones on the base space as $S^4(\frac{1}{2})$.

Proof. Note that each diagonal element with $\pm$ expressions combined has the parameter which varies on the 4-sphere identified with the connected sum of two of the 4-ball attached along their boundaries as the 3-sphere, as above. If one of three diagonal elements is fixed, then the situation is reduced to the 2×2 matrix case with compound order. Indeed, the signs in diagonal elements as well as trace values are determined by the condition (O_3) as the trace condition deduced below. $\square$

Remark. As a picture as the identification involved in the fiber spaces,

$$\begin{pmatrix} \pm & \leftarrow & \leftarrow \\ \uparrow & \pm & \leftarrow' \\ \uparrow & \uparrow' & \pm \end{pmatrix}, \quad \begin{pmatrix} \pm & \uparrow & \leftarrow' \\ \leftarrow & \pm & \leftarrow \\ \uparrow' & \uparrow & \pm \end{pmatrix}, \quad \begin{pmatrix} \pm & \leftarrow' & \leftarrow \\ \uparrow' & \pm & \leftarrow \\ \uparrow & \uparrow & \pm \end{pmatrix},$$

where each arrow, dashed or not, is identified with its transposed arrow, and the signs $\pm$ may correspond to upward arrows and leftward arrows, respectively.

Example 3.5. Suppose the bounded condition (B_3) defined above. Consider a few cases of taking limits to the boundary.

(a) The squared radius $|z_1|^2 + |z_2|^2$ for (z_1, z_2) goes to zero if and only if both upper bounds $\frac{1}{4} - |z_1|^2$ and $\frac{1}{4} - |z_2|^2$ for $|z_3|^2$ go to $\frac{1}{4}$. In this case,

$$\lim_{z_1, z_2 \to 0} a^+(z_1, z_2) = 1 \text{ or } \lim_{z_1, z_2 \to 0} a^-(z_1, z_2) = 0,$$

$$\lim_{z_1 \to 0} b^\pm(z_1, z_3) = \frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_3|^2} \quad \text{if } 0 \leq |z_3|^2 \leq \frac{1}{4},$$

$$\lim_{z_2 \to 0} c^\pm(z_2, z_3) = \frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_3|^2} \quad \text{if } 0 \leq |z_3|^2 \leq \frac{1}{4}.$$

Namely, it follows that the limit matrices are given by

$$\begin{pmatrix} 1 \text{ or } 0 & 0 & 0 \\ 0 & \frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_3|^2} & \overline{z_3} \\ 0 & z_3 & \frac{1}{2} \mp \sqrt{\frac{1}{4} - |z_3|^2} \end{pmatrix}$$

($\pm, \mp$ in compound order), which are generalized Bott projections in our sense, for $0 < |z_3|^2 \leq \frac{1}{4}$, with trace 2 or 1.

(b) Also, one of $|z_1|^2$ or $|z_2|^2$ goes to $\frac{1}{4}$ if and only if the other of $|z_1|^2$ or $|z_2|^2$ goes to zero and $|z_3|^2$ goes to zero. In these cases,

$$\lim_{|z_1| \text{ or } |z_2| \to 0 \text{ or } \frac{1}{4}} a^{\pm}(z_1, z_2) = \frac{1}{2},$$

$$\lim_{|z_1| \text{ or } |z_2| \to 0 \text{ or } \frac{1}{4}} b^{\pm}(z_1, z_3) = \frac{1}{2},$$

$$\lim_{|z_1| \text{ or } |z_2| \to 0 \text{ or } \frac{1}{4}} c^{\pm}(z_2, z_3) = \frac{1}{2},$$

and hence these cases do not happen, because the trace $\frac{3}{2}$ is not an integer. Such a partial limit to the boundary is not allowed, because the matrix size is odd, 3.

(c) Moreover, if $|z_1|^2 + |z_2|^2$ goes to $\frac{1}{4}$, then

$$\lim_{|z_1|^2+|z_2|^2 \to \frac{1}{4}} a^{\pm}(z_1, z_2) = \frac{1}{2} \quad \text{if} \quad \lim_{|z_1|^2+|z_2|^2 \to \frac{1}{4}} |z_1| \lim_{|z_1|^2+|z_2|^2 \to \frac{1}{4}} |z_2| \equiv w_1 w_2 \neq 0,$$

$$\lim_{|z_1|^2+|z_2|^2 \to \frac{1}{4}} b^{\pm}(z_1, z_3) = \frac{1}{2} \pm \sqrt{\frac{1}{4} - w_1^2 - |z_3|^2} \quad \text{if } 0 \leq |z_3|^2 \leq \frac{1}{4} - w_1^2,$$

$$\lim_{|z_1|^2+|z_2|^2 \to \frac{1}{4}} c^{\pm}(z_2, z_3) = \frac{1}{2} \pm \sqrt{\frac{1}{4} - w_2^2 - |z_3|^2} \quad \text{if } 0 \leq |z_3|^2 \leq \frac{1}{4} - w_2^2.$$

Together with the trace 1 or 2 given, the above conditions may determine the value $|z_3|$ by calculation.

Lemma 3.6. *Suppose that* p *is a generically non-degenerate projection of* $M_3(\mathbb{C})$. *If* p *has trace* 1, *then with* $\pm$ *in compound order,*

$$\overline{z_1} = \overline{z_1}^{\pm}(z_2, z_3) \equiv \frac{\overline{z_2} z_3}{\frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_2|^2 - |z_3|^2}}, \quad \textit{with } 0 < |z_2|^2 + |z_3|^2 \leq \frac{1}{4},$$

$$\overline{z_2} = \overline{z_2}^{\pm} \equiv \frac{\overline{z_1 z_3}}{\frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_1|^2 - |z_3|^2}}, \quad \textit{with } 0 < |z_1|^2 + |z_3|^2 \leq \frac{1}{4},$$

$$\overline{z_3} = \overline{z_3}^{\pm} \equiv \frac{z_1 \overline{z_2}}{\frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_1|^2 - |z_2|^2}}, \quad \textit{with } 0 < |z_1|^2 + |z_2|^2 \leq \frac{1}{4},$$

so that, as the radius condition $(R_{3,1})$,

$$|z_1| = |z_1|^{\pm} \equiv \frac{|z_2||z_3|}{\frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_2|^2 - |z_3|^2}}, \quad \text{with } 0 < |z_2|^2 + |z_3|^2 \leq \frac{1}{4},$$

$$|z_2| = |z_2|^{\pm} \equiv \frac{|z_1||z_3|}{\frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_1|^2 - |z_3|^2}}, \quad \text{with } 0 < |z_1|^2 + |z_3|^2 \leq \frac{1}{4},$$

$$|z_3| = |z_3|^{\pm} \equiv \frac{|z_1||z_2|}{\frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_1|^2 - |z_2|^2}}, \quad \text{with } 0 < |z_1|^2 + |z_2|^2 \leq \frac{1}{4}.$$

Note that for example, $z_1^+ = z_1^-$ *as well as* $|z_1|^+ = |z_1|^-$ *if and only if* $|z_2|^2 + |z_3|^2 = \frac{1}{4}$, *and so on.*

If p *has trace* 2, *then*

$$\overline{z_1} = \overline{z_1}^{\pm} = \frac{\overline{z_2}z_3}{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_2|^2 - |z_3|^2}}, \quad \text{with } 0 < |z_2|^2 + |z_3|^2 \leq \frac{1}{4},$$

$$\overline{z_2} = \overline{z_2}^{\pm} = \frac{\overline{z_1 z_3}}{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_1|^2 - |z_3|^2}}, \quad \text{with } 0 < |z_1|^2 + |z_3|^2 \leq \frac{1}{4},$$

$$\overline{z_3} = \overline{z_3}^{\pm} = \frac{z_1\overline{z_2}}{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_1|^2 - |z_2|^2}}, \quad \text{with } 0 < |z_1|^2 + |z_2|^2 \leq \frac{1}{4},$$

so that, as the radius condition $(R_{3,2})$, *in compound order,*

$$|z_1| = |z_1|^{\pm} = \frac{|z_2||z_3|}{\frac{1}{2} \mp \sqrt{\frac{1}{4} - |z_2|^2 - |z_3|^2}}, \quad \text{with } 0 < |z_2|^2 + |z_3|^2 \leq \frac{1}{4},$$

$$|z_2| = |z_2|^{\pm} = \frac{|z_1||z_3|}{\frac{1}{2} \mp \sqrt{\frac{1}{4} - |z_1|^2 - |z_3|^2}}, \quad \text{with } 0 < |z_1|^2 + |z_3|^2 \leq \frac{1}{4},$$

$$|z_3| = |z_3|^{\pm} = \frac{|z_1||z_2|}{\frac{1}{2} \mp \sqrt{\frac{1}{4} - |z_1|^2 - |z_2|^2}}, \quad \text{with } 0 < |z_1|^2 + |z_2|^2 \leq \frac{1}{4}.$$

Hence, the radius condition $(R_{3,1})$ *is the same as* $(R_{3,2})$, *up to signs* $\pm$ *in the formulae exchanged.*

Note also that for example, $z_1^+ = z_1^-$ *as well as* $|z_1|^+ = |z_1|^-$ *if and only if* $|z_2|^2 + |z_3|^2 = \frac{1}{4}$, *and so on.*

Therefore, the definition domains of those functions of two variables z_j *and* z_k $(1 \leq j < k \leq 3)$ *given above, such as* $\overline{z_1}^{\pm} = \overline{z_1}^{\pm}(z_2, z_3)$, *are given as the connected sum*

$$(B^4(\frac{1}{2}) \setminus \{0_4\}) \sqcup_{S^3(\frac{1}{2})} (B^2(\frac{1}{2}) \setminus \{0_4\})$$

of two copies of the real 4*-dimensional closed ball* $B^4(\frac{1}{2}) \setminus \{0_4\}$ *with radius* $\frac{1}{2}$ *and origin* 0_4 *removed, attached along the real* 3*-dimensional sphere* $S^3(\frac{1}{2})$ *with radius* $\frac{1}{2}$.

Proof. For instance, by the condition (O_3), note that

$$\overline{z_2}z_3 = (1-a-b)\overline{z_1} = (1+c-\mathrm{tr}(p))\overline{z_1}.$$

Then use the condition (D_3). □

Lemma 3.7. *In particular, suppose that $x, y, z > 0$ are positive reals, and that*

$$z = \frac{xy}{\frac{1}{2} \pm \sqrt{\frac{1}{4} - x^2 - y^2}} \quad \text{or} \quad z = \frac{xy}{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - x^2 - y^2}}$$

These equations are converted respectively, to

$$z^2 - \frac{xy}{x^2+y^2}z + \frac{x^2y^2}{x^2+y^2} = 0 \quad \text{or} \quad z^2 + \frac{xy}{x^2+y^2}z + \frac{x^2y^2}{x^2+y^2} = 0.$$

Moreover, by using the polar coordinate as $x = r\cos\theta$ and $y = r\sin\theta$ with $0 < r \le \frac{1}{2}$ and $0 < \theta < \frac{\pi}{2}$, those are converted respectively, to

$$z^2 - \frac{\sin 2\theta}{2}z + \frac{r^2\sin^2 2\theta}{4} = 0 \quad \text{or} \quad z^2 + \frac{\sin 2\theta}{2}z + \frac{r^2\sin^2 2\theta}{4} = 0.$$

Equivalently,

$$\left(z - \frac{\sin 2\theta}{4}\right)^2 = \frac{(1-4r^2)\sin^2 2\theta}{4^2} \quad \text{or} \quad \left(z + \frac{\sin 2\theta}{4}\right)^2 = \frac{(1-4r^2)\sin^2 2\theta}{4^2}.$$

Remark. It might be an interesting exercise (or problem) to draw the picture (or graph) determined by the equations above.

Proposition 3.8. *If p is a generically non-degenerate projection of $M_3(\mathbb{C})$, then the trace of p is given by, as the trace condition (T_3),*

$$\begin{aligned}\mathrm{tr}(p) = a+b+c &= \frac{1}{2}\left(3 - \frac{z_1\overline{z_2}z_3}{|z_1|^2} - \frac{\overline{z_1}z_2\overline{z_3}}{|z_2|^2} - \frac{z_1\overline{z_2}z_3}{|z_3|^2}\right) \\ &= \frac{3}{2} - \frac{1}{2}\left(\frac{1}{|z_1|^2} + \frac{1}{|z_2|^2} + \frac{1}{|z_3|^2}\right)\mathrm{Re}(z_1\overline{z_2}z_3)\end{aligned}$$

and

$$0 = \left(\frac{1}{|z_1|^2} - \frac{1}{|z_2|^2} + \frac{1}{|z_3|^2}\right)\mathrm{Im}(z_1\overline{z_2}z_3).$$

It $\mathrm{tr}(p)$ is 1, then $\mathrm{Re}(z_1\overline{z_2}z_3) > 0$.
It $\mathrm{tr}(p)$ is 2, then $\mathrm{Re}(z_1\overline{z_2}z_3) < 0$.
Moreover, in fact, $z_1\overline{z_2}z_3 \in \mathbb{R}$ since

$$z_1\overline{z_2}z_3 = \begin{cases} \frac{|z_1|^2|z_3|^2}{\frac{1}{2}\pm\sqrt{\frac{1}{4}-|z_1|^2-|z_3|^2}} > 0, & \text{if } \mathrm{tr}(p) = 1, \\ \frac{|z_1|^2|z_3|^2}{-\frac{1}{2}\pm\sqrt{\frac{1}{4}-|z_1|^2-|z_3|^2}} < 0, & \text{if } \mathrm{tr}(p) = 2. \end{cases}$$

Suppose that $z_j = |z_j|e^{i\theta_j}$ *for* $j = 1, 2, 3$, *with* $\theta_j \in [0, 2\pi]$. *It then follows that, as the angle condition* (A_3),

$$e^{i(\theta_1-\theta_2+\theta_3)} = \begin{cases} 1 & \text{if } \theta_1 - \theta_2 + \theta_3 = 0 \ \text{mod } 2\pi, \\ -1 & \text{if } \theta_1 - \theta_2 + \theta_3 = \pi \ \text{mod } 2\pi. \end{cases}$$

Equivalently, $e^{i\theta_2} = \pm e^{i(\theta_1+\theta_3)}$ *with* $\theta_1 + \theta_3 = \theta_2$ *or* $\theta_2 + \pi$ *mod* 2π.

Proof. It follows from the condition (O_3) that

$$(a+b) + \frac{\overline{z_2}z_3}{\overline{z_1}} = 1, \quad (a+c) + \frac{\overline{z_1}\,\overline{z_3}}{\overline{z_2}} = 1, \quad \frac{z_1\overline{z_2}}{\overline{z_3}} + (b+c) = 1.$$

Hence, adding both sides of these equations implies

$$\mathrm{tr}(p) = a + b + c = \frac{1}{2}\left(3 - \frac{z_1\overline{z_2}z_3}{|z_1|^2} - \frac{\overline{z_1}z_2\overline{z_3}}{|z_2|^2} - \frac{z_1\overline{z_2}z_3}{|z_3|^2}\right).$$

By adding and subtracting their complex conjugates on both sides we obtain the second and third equalities in the statement. These formulae are improved by using Lemma 3.6. □

Example 3.9. In particular, suppose that $z_1 = z_2 = z_3 = z \neq 0$. Then

$$\mathrm{tr}(p) = a + b + c = \frac{1}{2}(3 - 2z - \overline{z})$$

by using the first equation (T_3) of Proposition 3.8 above.

If $\mathrm{tr}(p) = 1$, then $z = \mathrm{Re}(z) = \frac{1}{3} = \overline{z}$. Then

$$p = p_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \equiv \begin{pmatrix} a^- = \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & b^- = \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & c^- = \frac{1}{3} \end{pmatrix}.$$

If $\mathrm{tr}(p) = 2$, then $z = \mathrm{Re}(z) = -\frac{1}{3} = \overline{z}$, with $|-\frac{1}{3}| = \frac{1}{3}$. Then

$$p = p_2(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}) \equiv \begin{pmatrix} a^+ = \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & b^+ = \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & c^+ = \frac{2}{3} \end{pmatrix}.$$

Proof. If $\mathrm{tr}(p) = 1$, then $1 = 2z + \overline{z}$ and $1 = 2\overline{z} + z$. It follows that $z = \mathrm{Re}(z) = \frac{1}{3} = \overline{z}$. Then

$$\begin{aligned} a = a^{\pm}(\frac{1}{3}, \frac{1}{3}) &= b = b^{\pm}(\frac{1}{3}, \frac{1}{3}) = c = c^{\pm}(\frac{1}{3}, \frac{1}{3}) \\ &= \frac{3 \pm 1}{6} = \frac{2}{3}, \frac{1}{3} \quad \text{with } 0 \leq \frac{2}{9} \leq \frac{1}{4} = \frac{2}{8}. \end{aligned}$$

Thus, $a = a^- = b = b^- = c = c^- = \frac{1}{3}$.

If $\mathrm{tr}(p) = 2$, then $-1 = 2z + \overline{z}$ and $-1 = 2\overline{z} + z$. It follows that $z = \mathrm{Re}(z) = -\frac{1}{3} = \overline{z}$. The same equations above also hold in this case. Hence $a = a^+ = b = b^+ = c = c^+ = \frac{3}{2}$. □

Example 3.10. Moreover, suppose that $|z_1| = |z_2| = |z_3| = r > 0$.

If p has trace either 1 or two, then by Lemma 3.6,

$$r = \frac{r^2}{\frac{1}{2} \pm \sqrt{\frac{1}{4} - 2r^2}}.$$

Hence $r = \frac{1}{3}$ Then $a^{\pm} = \frac{2}{3}, \frac{1}{3}$, $b^{\pm} = \frac{2}{3}, \frac{1}{3}$, and $c^{\pm} = \frac{2}{3}, \frac{1}{3}$, respectively.

Thus, if p has trace either 1 or 2, then

$$p = p_k(\frac{e^{i\theta_1}}{3}, \frac{e^{i\theta_2}}{3}, \frac{e^{i\theta_3}}{3}) = \begin{pmatrix} a^- = \frac{1}{3} \text{ or } a^+ = \frac{2}{3} & 3^{-1}e^{-i\theta_1} & 3^{-1}e^{-i\theta_2} \\ 3^{-1}e^{i\theta_1} & \frac{1}{3} \text{ or } \frac{2}{3} & 3^{-1}e^{-i\theta_3} \\ 3^{-1}e^{i\theta_2} & 3^{-1}e^{i\theta_3} & \frac{1}{3} \text{ or } \frac{2}{3} \end{pmatrix}$$

($k = 1$ or 2) respectively, where $\theta_1 - \theta_2 + \theta_3 = 0$ or π mod 2π by (A_3).

Example 3.11. Next suppose that $z_1 = z_2 = z \neq z_3 = w$ with $zw \neq 0$. Then

$$\operatorname{tr}(p) = a + b + c = \frac{1}{2}(3 - w - \overline{w} - \frac{|z|^2}{\overline{w}})$$

by using the first equation (T_3) of Proposition 3.8 above.

If $\operatorname{tr}(p) = 1$, then

$$w = w_{\pm}(|z|^2) \equiv \frac{1}{4} \pm \frac{1}{4}\sqrt{1 - 8|z|^2}, \quad \text{with } 0 < |z|^2 \leq \frac{1}{8},$$

so that $0 < w < \frac{1}{2}$. Note that $w_+(|z|^2) = w_-(|z|^2) = \frac{1}{4}$ if and only if $|z|^2 = \frac{1}{8}$, and the images of w_+ and w_- are $[\frac{1}{4}, \frac{1}{2})$ and $(0, \frac{1}{4}]$ respectively. Moreover,

$$p = p_1(z, z, w) \equiv \begin{pmatrix} a^- = 1 - 2w & \overline{z} & \overline{z} \\ z & b^+ = w & w \\ z & w & c^+ = w \end{pmatrix}.$$

Set $p_1^{\pm}(z, z, w^{\pm}) = p_1(z, z, w^{\pm})$. It follows that

$$\lim_{z \to 0} p_1^+(z, z, w^+) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad \lim_{z \to 0} p_1^-(z, z, w^-) = p_1 = 1 \oplus 0_2.$$

The space of the projections $p = p_1(z, z, w)$ with those parameters of z and w, together with these limit projections, all of which are identified with restrictive coordinates $(z, w) \in \mathbb{C} \times \mathbb{R}$, is homeomorphic to the 2-dimensional sphere S^2.

If $\operatorname{tr}(p) = 2$, then

$$w = w_{\pm}(|z|^2) \equiv -\frac{1}{4} \pm \frac{1}{4}\sqrt{1 - 8|z|^2}, \quad \text{with } 0 < |z|^2 \leq \frac{1}{8},$$

so that $-\frac{1}{2} < w < 0$. Note that $w_+(|z|^2) = w_-(|z|^2) = -\frac{1}{4}$ if and only if $|z|^2 = \frac{1}{8}$, and the images of w_+ and w_- are $[-\frac{1}{4}, 0)$ and $(-\frac{1}{2}, -\frac{1}{4}]$ respectively. Moreover,

$$p = p_2(z, z, w) \equiv \begin{pmatrix} a^- = -2w & \overline{z} & \overline{z} \\ z & b^+ = 1 + w & w \\ z & w & c^+ = 1 + w \end{pmatrix}.$$

Set $p_2^{\pm}(z,z,w^{\pm}) = p_2(z,z,w^{\pm})$. It follows that

$$\lim_{z\to 0} p_1^+(z,z,w^+) = 0 \oplus 1_2 \quad \text{and} \quad \lim_{z\to 0} p_1^-(z,z,w^-) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The space of the projections $p = p_2(z,z,w)$ with those parameters of z and w, together with these limit projections, all of which are identified with restrictive coordinates $(z,w) \in \mathbb{C} \times \mathbb{R}$, is homeomorphic to the 2-dimensional sphere S^2.

Proof. If $\mathrm{tr}(p) = 1$, then $1 = w + \overline{w} + \frac{|z|^2}{|w|^2} w$ and $1 = w + \overline{w} + \frac{|z|^2}{|w|^2}\overline{w}$. It follows that $1 = \mathrm{Re}(w)(2 + \frac{|z|^2}{|w|^2})$ and $0 = \mathrm{Im}(w)$, so that $w = \overline{w}$. The equation $1 = w(2 + \frac{|z|^2}{w^2})$ with $w \in \mathbb{R}$ is converted to $|z|^2 = w - 2w^2 = w(1-2w)$, and hence $0 \le w \le \frac{1}{2}$. Solving this equation we obtain the equality for w in the statement. Then

$$\begin{aligned} a = a^{\pm}(z,z) &= \frac{1}{2} \pm \sqrt{\frac{1}{4} - 2|z|^2} \quad \text{with } 0 < |z|^2 \le \frac{1}{8}, \\ &= 2w, 1 - 2w, \\ b = b^{\pm}(z,w) &= \frac{1}{2} \pm (w - \frac{1}{2}) = w, -w+1, \\ c = c^{\pm}(z,w) &= \frac{1}{2} \pm (w - \frac{1}{2}) = w, -w+1. \end{aligned}$$

Therefore, the admissible signs of the diagonal elements are $(-,+,+)$.

If $\mathrm{tr}(p) = 2$, then $-1 = w + \overline{w} + \frac{|z|^2}{|w|^2} w$ and $-1 = w + \overline{w} + \frac{|z|^2}{|w|^2}\overline{w}$. It follows that $-1 = \mathrm{Re}(w)(2 + \frac{|z|^2}{|w|^2})$ and $w = \overline{w}$. The equation $-1 = w(2 + \frac{|z|^2}{w^2})$ with $w \in \mathbb{R}$ is converted to $|z|^2 = -w - 2w^2 = -w(1+2w)$, and hence $-\frac{1}{2} \le w \le 0$. Solving this equation we obtain the equality for w in the statement. Then

$$\begin{aligned} a = a^{\pm}(z,z) &= \frac{1}{2} \pm \sqrt{\frac{1}{4} - 2|z|^2} \quad \text{with } 0 < |z|^2 \le \frac{1}{8}, \\ &= 1 + 2w, -2w, \\ b = b^{\pm}(z,w) &= \frac{1}{2} \pm (w + \frac{1}{2}) = 1 + w, -w, \\ c = c^{\pm}(z,w) &= \frac{1}{2} \pm (w + \frac{1}{2}) = 1 + w, -w. \end{aligned}$$

Therefore, the admissible signs of the diagonal elements are $(-,+,+)$. □

Example 3.12. Moreover, suppose that $|z_1| = |z_2| = r \ne |z_3| = r'$ with $r'r \ne 0$. Then, by Lemma 3.6,

$$r' = r^2 \left(\frac{1}{2} \pm \sqrt{\frac{1}{4} - 2r^2} \right)^{-1}.$$

By Lemma 3.1,

$$\begin{aligned} a^{\pm} &= \frac{1}{2} \pm \sqrt{4^{-1} - 2r^2}, \quad \text{with } 0 < r^2 \le \frac{1}{8}, \\ b^{\pm} &= \frac{1}{2} \pm \sqrt{4^{-1} - r^2 - (r')^2}, \quad \text{with } 0 < r^2 + (r')^2 \le \frac{1}{4}, \\ c^{\pm} &= \frac{1}{2} \pm \sqrt{4^{-1} - r^2 - (r')^2}, \quad \text{with } 0 < r^2 + (r')^2 \le \frac{1}{4}. \end{aligned}$$

Example 3.13. Similarly, suppose that $z_1 = w \neq z_2 = z_3 = z$ with $zw \neq 0$. Then

$$\mathrm{tr}(p) = a + b + c = \frac{1}{2}(3 - \frac{|z|^2}{\overline{w}} - \overline{w} - w)$$

by using the first equation (T_3) of Proposition 3.8 above. Note that this equation is the same as in the example above.

If $\mathrm{tr}(p) = 1$, then

$$w = w_{\pm}(|z|^2) \equiv \frac{1}{4} \pm \frac{1}{4}\sqrt{1 - 8|z|^2}, \quad \text{with } 0 < |z|^2 \le \frac{1}{8},$$

so that $0 < w < \frac{1}{2}$. Moreover,

$$p = p_1(w, z, z) \equiv \begin{pmatrix} a^+ = w & w & \overline{z} \\ w & b^+ = w & \overline{z} \\ z & z & c^- = 1 - 2w \end{pmatrix}.$$

Set $p_1^{\pm}(w^{\pm}, z, z) = p_1(w^{\pm}, z, z)$. It follows that

$$\lim_{z \to 0} p_1^+(w^+, z, z) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \lim_{z \to 0} p_1^-(w^-, z, z) = p_3 = 0_2 \oplus 1.$$

The space of the projections $p = p_1(w, z, z)$ with those parameters of z and w, together with these limit projections, all of which are identified with restrictive coordinates $(z, w) \in \mathbb{C} \times \mathbb{R}$, is homeomorphic to the 2-dimensional sphere S^2.

If $\mathrm{tr}(p) = 2$, then

$$w = w_{\pm}(|z|^2) \equiv -\frac{1}{4} \pm \frac{1}{4}\sqrt{1 - 8|z|^2}, \quad \text{with } 0 < |z|^2 \le \frac{1}{8},$$

so that $-\frac{1}{2} < w < 0$. Moreover,

$$p = p_2(w, z, z) \equiv \begin{pmatrix} a^+ = 1 + w & w & \overline{z} \\ w & b^+ = 1 + w & \overline{z} \\ z & z & c^- = -2w \end{pmatrix}.$$

Set $p_2^{\pm}(w^{\pm}, z, z) = p_2(w^{\pm}, z, z)$. It follows that

$$\lim_{z \to 0} p_1^+(w^+, z, z) = 1_2 \oplus 0 \quad \text{and} \quad \lim_{z \to 0} p_1^-(w^-, z, z) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The space of the projections $p = p_2(w, z, z)$ with those parameters of z and w, together with these limit projections, all of which are identified with restrictive coordinates $(z, w) \in \mathbb{C} \times \mathbb{R}$, is homeomorphic to the 2-dimensional sphere S^2.

Proof. If $\mathrm{tr}(p) = 1$, then $1 = w + \overline{w} + \frac{|z|^2}{|w|^2} w$ and $1 = w + \overline{w} + \frac{|z|^2}{|w|^2} \overline{w}$. It follows that $1 = \mathrm{Re}(w)(2 + \frac{|z|^2}{|w|^2})$ and $w = \overline{w}$. The equation $1 = w(2 + \frac{|z|^2}{w^2})$ with $w \in \mathbb{R}$ is converted to $|z|^2 = w - 2w^2 = w(1 - 2w)$, and hence $0 \le w \le \frac{1}{2}$. Solving this equation we obtain the equality for w in the statement. Then

$$\begin{aligned} a = a^{\pm}(w, z) &= \frac{1}{2} \pm (w - \frac{1}{2}) = w, -w + 1, \\ b = b^{\pm}(w, z) &= \frac{1}{2} \pm (w - \frac{1}{2}) = w, -w + 1, \\ c = c^{\pm}(z, z) &= \frac{1}{2} \pm \sqrt{\frac{1}{4} - 2|z|^2} \quad \text{with } 0 < |z|^2 \le \frac{1}{8}, \\ &= 2w, 1 - 2w. \end{aligned}$$

Therefore, the admissible signs of the diagonal elements are $(+, +, -)$.

If $\mathrm{tr}(p) = 2$, then $-1 = w + \overline{w} + \frac{|z|^2}{|w|^2} w$ and $-1 = w + \overline{w} + \frac{|z|^2}{|w|^2} \overline{w}$. It follows that $-1 = \mathrm{Re}(w)(2 + \frac{|z|^2}{|w|^2})$ and $w = \overline{w}$. The equation $-1 = w(2 + \frac{|z|^2}{w^2})$ with $w \in \mathbb{R}$ is converted to $|z|^2 = -w - 2w^2 = -w(1 + 2w)$, and hence $-\frac{1}{2} \le w \le 0$. Solving this equation we obtain the equality for w in the statement. Then

$$\begin{aligned} a = a^{\pm}(w, z) &= \frac{1}{2} \pm (w + \frac{1}{2}) = 1 + w, -w, \\ b = b^{\pm}(w, z) &= \frac{1}{2} \pm (w + \frac{1}{2}) = 1 + w, -w, \\ c = c^{\pm}(z, z) &= \frac{1}{2} \pm \sqrt{\frac{1}{4} - 2|z|^2} \quad \text{with } 0 < |z|^2 \le \frac{1}{8}, \\ &= 1 + 2w, -2w. \end{aligned}$$

Therefore, the admissible signs of the diagonal elements are $(+, +, -)$. □

Example 3.14. Furthermore, suppose that $z_1 = z_3 = z \ne z_2 = w$ with $zw \ne 0$. Then

$$\mathrm{tr}(p) = a + b + c = \frac{1}{2}(3 - \frac{2z^2}{|z|^2}\overline{w} - \frac{(\overline{z})^2}{|w|^2} w)$$

by using the first equation (T_3) of Proposition 3.8 above. Note that this equation is the same as in the example above.

If $\mathrm{tr}(p) = 1$, then

$$w = w^{\pm}(z) = \frac{z^2}{\frac{1}{2} \pm \sqrt{\frac{1}{4} - 2|z|^2}}, \quad \text{with } 0 < |z|^2 \le \frac{1}{8}$$

with $\lim_{|z|\to 0} w^+(z) = 0$ and $\lim_{|z|\to 0} |w|^-(z) = \frac{1}{2}$. Moreover,

$$p = p_1(z, w, z) \equiv \begin{pmatrix} a^- & z & \overline{w} \\ z & b^- & \overline{z} \\ w & z & c^- \end{pmatrix}.$$

Set $p_1^\pm(z, w^\pm, z) = p_1(z, w^\pm, z)$. It follows that

$$\lim_{z\to 0} p_1^+(z, w^+, z) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad \lim_{z\to 0} p_1^-(z, w^-, z) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

The space of the projections $p = p_1(z, w, z)$ with those parameters of z and w, together with these limit projections, all of which are identified with restrictive coordinates $(z, z^2\overline{w}) \in \mathbb{C} \times \mathbb{R}$, is homeomorphic to the 2-dimensional sphere S^2.

If $\mathrm{tr}(p) = 2$, then

$$w = w^\pm(z) = \frac{z^2}{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 2|z|^2}}, \quad \text{with } 0 < |z|^2 \le \frac{1}{8}$$

with $\lim_{|z|\to 0} |w|^+(z) = \frac{1}{2}$ and $\lim_{|z|\to 0} w^-(z) = 0$. Moreover,

$$p = p_2(z, w, z) \equiv \begin{pmatrix} a^- & z & \overline{w} \\ z & b^+ & \overline{z} \\ w & z & c^- \end{pmatrix}.$$

Set $p_2^\pm(z, w^\pm, z) = p_2(z, w^\pm, z)$. It follows that

$$\lim_{z\to 0} p_1^+(z, w^+, z) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}, \quad \lim_{z\to 0} p_1^-(z, w^-, z) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

The space of the projections $p = p_2(z, w, z)$ with those parameters of z and w, together with these limit projections, all of which are identified with restrictive coordinates $(z, z^2\overline{w}) \in \mathbb{C} \times \mathbb{R}$, is homeomorphic to the 2-dimensional sphere S^2.

Proof. If $\mathrm{tr}(p) = 1$, then $1 = \frac{2z^2}{|z|^2}\overline{w} + \frac{(\overline{z})^2}{|w|^2}w$ and $1 = \frac{2(\overline{z})^2}{|z|^2}w + \frac{z^2}{|w|^2}\overline{w}$. It follows that $1 = \mathrm{Re}(z^2\overline{w})(\frac{2}{|z|^2} + \frac{1}{|w|^2})$ and $0 = (\frac{2}{|z|^2} - \frac{1}{|w|^2})\mathrm{Im}(z^2\overline{w})$. On the other hand, by Lemma 3.6,

$$z^2\overline{w} = z\overline{w}z = \frac{|z|^4}{\frac{1}{2} \pm \sqrt{\frac{1}{4} - 2|z|^2}}, \quad \text{with } 0 < |z|^2 \le \frac{1}{8}.$$

Hence,

$$w = w^\pm(z) \equiv \frac{z^2}{\frac{1}{2} \pm \sqrt{\frac{1}{4} - 2|z|^2}}, \quad \text{with } 0 < |z|^2 \le \frac{1}{8}.$$

Thus,

$$|w| = |w|^{\pm}(z) \equiv \frac{|z|^2}{\frac{1}{2} \pm \sqrt{\frac{1}{4} - 2|z|^2}}, \quad \text{with } 0 < |z|^2 \leq \frac{1}{8}.$$

It follows that $\lim_{|z|\to 0} |w|^{+}(z) = 0$, but

$$\lim_{|z|\to 0} |w|^{-}(z) = \lim_{|z|\to 0} \frac{2|z|}{-\frac{1}{2}(\frac{1}{4} - 2|z|^2)^{-\frac{1}{2}}(-4|z|)} = \frac{1}{2}$$

by the l'Hospital theorem.

Then

$$\begin{aligned} a = a^{\pm}(z,w) &= \frac{1}{2} \pm \sqrt{\frac{1}{4} - |z|^2 - |w|^2} \\ b = b^{\pm}(z,z) &= \frac{1}{2} \pm \sqrt{\frac{1}{4} - 2|z|^2} \quad \text{with } 0 < |z|^2 \leq \frac{1}{8}, \\ &= \frac{|z|^4}{(\overline{z})^2 w} = \frac{z^2}{w}, \\ c = c^{\pm}(w,z) &= \frac{1}{2} \pm \sqrt{\frac{1}{4} - |w|^2 - |z|^2}. \end{aligned}$$

If we take the limit as $|z| \to 0$, then

$$\begin{aligned} \lim_{|z|\to 0} a^{+}(z,w) = 1, &\quad \lim_{|z|\to 0} a^{-}(z,w) = \frac{1}{2}, \\ \lim_{|z|\to 0} b^{+}(z,w) = 1, &\quad \lim_{|z|\to 0} b^{-}(z,w) = 0, \\ \lim_{|z|\to 0} c^{+}(z,w) = 1, &\quad \lim_{|z|\to 0} c^{-}(z,w) = \frac{1}{2}. \end{aligned}$$

Consequently, we could determine the signs as $(-,-,-)$ as in the statement.

If $\mathrm{tr}(p) = 2$, then $-1 = \frac{2z^2}{|z|^2}\overline{w} + \frac{(\overline{z})^2}{|w|^2}w$ and $-1 = \frac{2(\overline{z})^2}{|z|^2}w + \frac{z^2}{|w|^2}\overline{w}$. It follows that $-1 = \mathrm{Re}(z^2\overline{w})(\frac{2}{|z|^2} + \frac{1}{|w|^2})$ and $0 = (\frac{2}{|z|^2} - \frac{1}{|w|^2})\mathrm{Im}(z^2\overline{w})$. On the other hand, by Lemma 3.6,

$$z^2\overline{w} = z\overline{w}z = \frac{|z|^4}{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 2|z|^2}}, \quad \text{with } 0 < |z|^2 \leq \frac{1}{8}.$$

Hence,

$$w = w^{\pm}(z) \equiv \frac{z^2}{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 2|z|^2}}, \quad \text{with } 0 < |z|^2 \leq \frac{1}{8}.$$

Thus, with compound order,

$$|w| = |w|^{\pm}(z) \equiv \frac{|z|^2}{\frac{1}{2} \mp \sqrt{\frac{1}{4} - 2|z|^2}}, \quad \text{with } 0 < |z|^2 \leq \frac{1}{8}.$$

It follows that $\lim_{|z|\to 0} |w|^+(z) = \frac{1}{2}$ and $\lim_{|z|\to 0} |w|^-(z) = 0$, as shown above. Similarly, as in the case where the trace is one, we can obtain the similar conclusion with signs $(-,+,-)$ as in the statement. □

Table 2: The involved signs, only points, or limits at zeros (and not) of the diagonal components $(a^\pm, b^\pm, c^\pm)$ with the off diagonal conditions limited

Trace	Sign	Diagonal	Off diagonal
0	$(-,-.-)$	Point $(0,0,0)$	$z_j = 0\ (1 \leq j \leq 3)$
1	$(-,-.-)$	Point $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$z_j = z \neq 0, = \frac{1}{3}$
	$(-,-.-)$	Point $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$\|z_j\| = r > 0, = \frac{1}{3}$
	$(-,+,+)$	Limits $(0,\frac{1}{2},\frac{1}{2})^{(+)}$, $(1,0,0)^{(-)}$ as $z_1 \to 0$, $z_3^+ \to \frac{1}{2}$, $z_3^- \to 0$	$z_1 = z_2 \neq z_3^{(\pm)}$, $z_1 z_3 \neq 0$
	$(+,+,-)$	Limits $(\frac{1}{2},\frac{1}{2},0)^{(+)}$, $(0,0,1)^{(-)}$ as $z_2 \to 0$, $z_1^+ \to \frac{1}{2}$, $z_1^- \to 0$	$z_1^{(\pm)} \neq z_2 = z_3$, $z_1 z_3 \neq 0$
	$(-,-,-)$	The same limit $(\frac{1}{2},0,\frac{1}{2})^{(\pm)}$ as $z_1 \to 0$, $z_2^+ \to 0$, $z_2^- \to \frac{1}{2}$	$z_1 = z_3 \neq z_2^{(\pm)}$, $z_1 z_2 \neq 0$
	?	?	$(z_i \neq z_j,\ z_1 z_2 z_3 \neq 0)$
2	$(+,+,+)$	Point $(\frac{2}{3},\frac{2}{3},\frac{2}{3})$	$z_j = z \neq 0, = -\frac{1}{3}$
	$(+,+,+)$	Point $(\frac{2}{3},\frac{2}{3},\frac{2}{3})$	$\|z_j\| = r > 0, = \frac{1}{3}$
	$(-,+,+)$	Limits $(0,1,1)^{(+)}$, $(1,\frac{1}{2},\frac{1}{2})^{(-)}$ as $z_1 \to 0$, $z_3^+ \to 0$, $z_3^- \to -\frac{1}{2}$	$z_1 = z_2 \neq z_3^{(\pm)}$, $z_1 z_3 \neq 0$
	$(+,+,-)$	Limits $(1,1,0)^{(+)}$, $(\frac{1}{2},\frac{1}{2},1)^{(-)}$ as $z_2 \to 0$, $z_1^+ \to 0$, $z_1^- \to -\frac{1}{2}$	$z_1^{(\pm)} \neq z_2 = z_3$, $z_1 z_3 \neq 0$
	$(-,+,-)$	The same limit $(\frac{1}{2},1,\frac{1}{2})^{(\pm)}$ as $z_1 \to 0$, $z_2^+ \to \frac{1}{2}$, $z_2^- \to 0$	$z_1 = z_3 \neq z_2^{(\pm)}$, $z_1 z_2 \neq 0$
	?	?	$(z_i \neq z_j,\ z_1 z_2 z_3 \neq 0)$
3	$(+,+.+)$	Point $(1,1,1)$	$z_1 = z_2 = z_3 = 0$

Remark. Note that the full degenerate case and the generically non-degenerate cases are contained in the table above, but not are the partially degenerate cases. The most generic case with lower off diagonal components not mutually identified as round bracketed as $(\cdots)$ with questions is certainly reduced to the partially less generic cases with some lower off diagonal components identified as above as suitable limits. Possibly, the most generic case is also understandable algebraically and geometrically by determining the signs involved in the diagonal components as well as the corresponding limits with respect to off diagonal components, as done above, but are not considered this time by the deadline, and should be considered in the possible section in the future.

4 The 4×4 matrix case

Solving the equation for projections we obtain

Lemma 4.1. *If $p = (p_{ij})$ is a projection of $M_4(\mathbb{C})$, then*

$$p = \begin{pmatrix} a_{11} & \overline{z_{21}} & \overline{z_{31}} & \overline{z_{41}} \\ z_{21} & a_{22} & \overline{z_{32}} & \overline{z_{42}} \\ z_{31} & z_{32} & a_{33} & \overline{z_{43}} \\ z_{41} & z_{42} & z_{43} & a_{44} \end{pmatrix}$$

with $a_{jj} \in \mathbb{R}$ for $1 \leq j \leq 4$, $z_{ji} \in \mathbb{C}$ for $1 \leq i < j \leq 4$, where

$$\begin{aligned} a_{11}^2 + |z_{21}|^2 + |z_{31}|^2 + |z_{41}|^2 &= a_{11}, \\ |z_{21}|^2 + a_{22}^2 + |z_{32}|^2 + |z_{42}|^2 &= a_{22}, \\ |z_{31}|^2 + |z_{32}|^2 + a_{33}^2 + |z_{43}|^2 &= a_{33}, \\ |z_{41}|^2 + |z_{42}|^2 + |z_{43}|^2 + a_{44} &= a_{44}, \end{aligned}$$

as the diagonal condition (D_4), and

$$\begin{aligned} &(a_{11} + a_{22})\overline{z_{21}} + \overline{z_{31}}z_{32} + \overline{z_{41}}z_{42} = \overline{z_{21}}, \quad (a_{11} + a_{33})\overline{z_{31}} + \overline{z_{21}}\overline{z_{32}} + \overline{z_{41}}z_{43} = \overline{z_{31}}, \\ &(a_{22} + a_{33})\overline{z_{32}} + z_{21}\overline{z_{31}} + \overline{z_{42}}z_{43} = \overline{z_{32}}, \quad (a_{11} + a_{44})\overline{z_{41}} + \overline{z_{21}}\overline{z_{42}} + \overline{z_{31}}\overline{z_{43}} = \overline{z_{41}}, \\ &(a_{22} + a_{44})\overline{z_{42}} + z_{21}\overline{z_{41}} + \overline{z_{32}}\overline{z_{43}} = \overline{z_{42}}, \quad (a_{33} + a_{44})\overline{z_{43}} + z_{31}\overline{z_{41}} + z_{32}\overline{z_{42}} = \overline{z_{43}}, \end{aligned}$$

as the off diagonal condition (O_4), so that the condition (D_4) implies that

$$\begin{aligned} a_{11} = a_{11}^{\pm}(z_{21}, z_{31}, z_{41}) &\equiv \frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_{21}|^2 - |z_{31}|^2 - |z_{41}|^2} \\ &\quad \textit{if } 0 \leq |z_{21}|^2 + |z_{31}|^2 + |z_{41}|^2 \leq \frac{1}{4}, \\ a_{22} = a_{22}^{\pm}(z_{21}, z_{32}, z_{42}) &\equiv \frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_{21}|^2 - |z_{32}|^2 - |z_{42}|^2} \\ &\quad \textit{if } 0 \leq |z_{21}|^2 + |z_{32}|^2 + |z_{42}|^2 \leq \frac{1}{4}, \\ a_{33} = a_{33}^{\pm}(z_{31}, z_{32}, z_{43}) &\equiv \frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_{31}|^2 - |z_{32}|^2 - |z_{43}|^2} \\ &\quad \textit{if } 0 \leq |z_{31}|^2 + |z_{32}|^2 + |z_{43}|^2 \leq \frac{1}{4}, \\ a_{44} = a_{44}^{\pm}(z_{41}, z_{42}, z_{43}) &\equiv \frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_{41}|^2 - |z_{42}|^2 - |z_{43}|^2} \\ &\quad \textit{if } 0 \leq |z_{41}|^2 + |z_{42}|^2 + |z_{43}|^2 \leq \frac{1}{4}, \end{aligned}$$

where the 3 variables of each diagonal component a_{jj} are off diagonal, components of each column vector of p containing a_{jj}.

Moreover, as the equality condition (E_4), $a_{11}^+ = a_{11}^-$ *if and only if* $|z_{21}|^2 + |z_{31}|^2 + |z_{41}|^2 = \frac{1}{4}$, *and* $\cdots$, *and* $a_{44}^+ = a_{44}^-$ *if and only if* $|z_{41}|^2 + |z_{42}|^2 + |z_{43}|^2 = \frac{1}{4}$.

Corollary 4.2. *It follows from Lemma* 4.1 *the following*:

(1) *If all* $z_{ji} = 0$, *then* $a_{jj} = 0$ *or* 1 *for* $1 \leq j \leq 4$.

(2) *If only one* $z_{ji} \neq 0$ *and the other all zero, then these correspond to the* 2×2 *case or the split* 2×2 *case.*

(3) *There are also the cases by choosing two split or not, blocks of the split or not,* 2×2 *cases.*

(4) *There is the rest of the other cases.*

We may say that a projection $p \in M_4(\mathbb{C})$ is **full degenerate** if it is in the case (1), and it is **partially degenerate** or a **generalized Bott** projection if in the cases (2) or (3), and it is **partially degenerate** and **not** a generalized Bott projection if in the case (4) but there is a zero component in the off diagonal part, and **generically non-degenerate** if in the case (4) and there are no zero components in the off diagonal part.

For convenience, we mainly consider the generically non-degenerate projections in what follows. The case of partially degenerate projections except generalized Bott projections may be considered elsewhere in the future.

Lemma 4.3. *As the bounded condition* (B_4), *in Lemma* 4.1, *if* $z_{21}, z_{31}, z_{41} \in \mathbb{C}$ *are fixed with* $0 \leq |z_{21}|^2 + |z_{31}|^2 + |z_{41}|^2 \leq \frac{1}{4}$, *then*

$$|z_{32}|^2 + |z_{42}|^2 \leq \frac{1}{4} - |z_{21}|^2,$$
$$|z_{32}|^2 + |z_{43}|^2 \leq \frac{1}{4} - |z_{31}|^2,$$
$$|z_{42}|^2 + |z_{43}|^2 \leq \frac{1}{4} - |z_{41}|^2,$$

and moreover, if z_{32} *and* z_{42} *are fixed with* $|z_{32}|^2 + |z_{42}|^2 \leq \frac{1}{4} - |z_{21}|^2$, *then*

$$|z_{43}|^2 \leq \min\{\frac{1}{4} - |z_{31}|^2 - |z_{32}|^2, \frac{1}{4} - |z_{41}|^2 - |z_{42}|^2\}.$$

Proposition 4.4. *Each connected component of the space* $P(M_4(\mathbb{C}))^\sim$ *is contained in the path-connected bounded set* $S^{6,4,2}$ *defined by the diagonal condition* (D_4) *as the bounded condition* (B_4), *which is homeomorphic to a fiber space over the base space* $S^6(\frac{1}{2})$ *identified with* $B^6(\frac{1}{2}) \sqcup_{S^5(\frac{1}{2})} B^6(\frac{1}{2})$, *with fibers obtained as fiber spaces* $S^{4,2}$ *over the base space* $S^4(\sqrt{\frac{1}{4} - r_1^2})$ *identified with*

$$B^4(\sqrt{\frac{1}{4} - r_1^2}) \sqcup_{S^3(\sqrt{\frac{1}{4} - r_1^2})} B^4(\sqrt{\frac{1}{4} - r_1^2}),$$

with fibers $S^2(\sqrt{\frac{1}{4} - r_1^2 - r_2^2})$ *identified with*

$$B^2(\sqrt{\frac{1}{4} - r_1^2 - r_2^2}) \sqcup_{S^1(\sqrt{\frac{1}{4} - r_1^2 - r_2^2})} B^2(\sqrt{\frac{1}{4} - r_1^2 - r_2^2}),$$

where the positive real parameters r_1 *and* r_2 *vary continuously, with* $0 \leq r_1^2 \leq \frac{1}{4}$ *and* $0 \leq r_1^2 + r_2^2 \leq \frac{1}{4}$*. Namely,*

$$\begin{array}{ccccc} S^{6,4,2} & \longleftarrow & S^{4,2} & \longleftarrow & S^2(\sqrt{\frac{1}{4} - r_1^2 - r_2^2}) \\ \downarrow & & \downarrow & & \\ S^6(\frac{1}{2}) & & S^4(\sqrt{\frac{1}{4} - r_1^2}) & & \end{array}$$

Proof. Note that each diagonal element with $\pm$ expressions combined has the parameter which varies on the 6-sphere identified with the connected sum of two of the 6-ball attached along their boundaries as the 5-sphere, as above. If one of 4 diagonal elements is fixed, then the situation is reduced to the 3×3 matrix case with compound orders. Indeed, the signs in diagonal elements as well as trace values are determined by the condition (O_4) as the trace condition deduced below. □

Lemma 4.5. *Suppose that* p *is a generically non-degenerate projection of* $M_4(\mathbb{C})$*.*
If p *has trace* k *with* $1 \leq k \leq 3$*, then for example,*

$$\overline{z_{21}} = \frac{\overline{z_{31}} z_{32} + \overline{z_{41}} z_{42}}{2 - k \pm \sqrt{\frac{1}{4} - |z_{31}|^2 - |z_{32}|^2 - |z_{43}|^2} \pm \sqrt{\frac{1}{4} - |z_{41}|^2 - |z_{42}|^2 - |z_{43}|^2}}$$

$$\text{with } 0 < |z_{32}|^2 + |z_{32}|^2 + |z_{43}|^2 \leq \frac{1}{4}, 0 < |z_{41}|^2 + |z_{42}|^2 + |z_{43}|^2 \leq \frac{1}{4},$$

$$\overline{z_{31}} = \frac{\overline{z_{21}}\,\overline{z_{32}} + \overline{z_{41}} z_{43}}{2 - k \pm \sqrt{\frac{1}{4} - |z_{21}|^2 - |z_{32}|^2 - |z_{42}|^2} \pm \sqrt{\frac{1}{4} - |z_{41}|^2 - |z_{42}|^2 - |z_{43}|^2}}$$

$$\text{with } 0 < |z_{21}|^2 + |z_{32}|^2 + |z_{42}|^2 \leq \frac{1}{4}, 0 < |z_{41}|^2 + |z_{42}|^2 + |z_{43}|^2 \leq \frac{1}{4},$$

$$\overline{z_{32}} = \frac{z_{21}\overline{z_{32}} + \overline{z_{42}} z_{43}}{2 - k \pm \sqrt{\frac{1}{4} - |z_{21}|^2 - |z_{31}|^2 - |z_{41}|^2} \pm \sqrt{\frac{1}{4} - |z_{41}|^2 - |z_{42}|^2 - |z_{43}|^2}}$$

$$\text{with } 0 < |z_{21}|^2 + |z_{31}|^2 + |z_{41}|^2 \leq \frac{1}{4}, 0 < |z_{41}|^2 + |z_{42}|^2 + |z_{43}|^2 \leq \frac{1}{4},$$

($\pm, \pm$ *in compound order), so that, as the radius condition* $(R_{4,k})$*,*

$$|z_{21}| = \frac{|\overline{z_{31}} z_{32} + \overline{z_{41}} z_{42}|}{2 - k \pm \sqrt{\frac{1}{4} - |z_{31}|^2 - |z_{32}|^2 - |z_{43}|^2} \pm \sqrt{\frac{1}{4} - |z_{41}|^2 - |z_{42}|^2 - |z_{43}|^2}},$$

with

$$|\overline{z_{31}} z_{32} + \overline{z_{41}} z_{42}| = \sqrt{|z_{31}|^2 |z_{32}|^2 + 2\,\mathrm{Re}(\overline{z_{31}} z_{32} \overline{z_{41}} z_{42}) + |z_{41}|^2 |z_{42}|^2}.$$

Proof. For example,

$$\overline{z_{31}} z_{32} + \overline{z_{41}} z_{42} = (1 - a_{11} - a_{22})\overline{z_{21}} = (1 + a_{33} + a_{44} - k)\overline{z_{21}}.$$

□

Proposition 4.6. *If p is a generically non-degenerate projection of $M_4(\mathbb{C})$, then the trace of p is given by, as the trace condition (T_4),*

$$\begin{aligned}
\mathrm{tr}(p) = \sum_{j=1}^{4} a_{jj} &= \frac{4}{2} - \frac{z_{21}\overline{z_{31}}z_{32} + z_{21}\overline{z_{41}}z_{42}}{3|z_{21}|^2} - \frac{\overline{z_{21}}z_{31}\overline{z_{32}} + z_{31}\overline{z_{41}}z_{43}}{3|z_{31}|^2} \\
&- \frac{\overline{z_{21}}z_{41}\overline{z_{42}} + \overline{z_{31}}z_{41}\overline{z_{43}}}{3|z_{41}|^2} - \frac{z_{21}\overline{z_{31}}z_{32} + z_{32}\overline{z_{42}}z_{43}}{3|z_{32}|^2} \\
&- \frac{z_{21}\overline{z_{41}}z_{42} + \overline{z_{32}}z_{42}\overline{z_{43}}}{3|z_{42}|^2} - \frac{z_{31}\overline{z_{41}}z_{43} + z_{32}\overline{z_{42}}z_{43}}{3|z_{43}|^2} \\
&= \frac{4}{2} - \frac{1}{3}\left(\frac{1}{|z_{21}|^2} + \frac{1}{|z_{31}|^2} + \frac{1}{|z_{32}|^2}\right)\mathrm{Re}(z_{21}\overline{z_{31}}z_{32}) \\
&- \frac{1}{3}\left(\frac{1}{|z_{21}|^2} + \frac{1}{|z_{41}|^2} + \frac{1}{|z_{42}|^2}\right)\mathrm{Re}(z_{21}\overline{z_{41}}z_{42}) \\
&- \frac{1}{3}\left(\frac{1}{|z_{31}|^2} + \frac{1}{|z_{41}|^2} + \frac{1}{|z_{43}|^2}\right)\mathrm{Re}(z_{31}\overline{z_{41}}z_{43}) \\
&- \frac{1}{3}\left(\frac{1}{|z_{32}|^2} + \frac{1}{|z_{42}|^2} + \frac{1}{|z_{43}|^2}\right)\mathrm{Re}(z_{32}\overline{z_{42}}z_{43})
\end{aligned}$$

and

$$\begin{aligned}
0 = &\left(\frac{1}{|z_{21}|^2} - \frac{1}{|z_{31}|^2} + \frac{1}{|z_{32}|^2}\right)\mathrm{Im}(z_{21}\overline{z_{31}}z_{32}) \\
+ &\left(\frac{1}{|z_{21}|^2} - \frac{1}{|z_{41}|^2} + \frac{1}{|z_{42}|^2}\right)\mathrm{Im}(z_{21}\overline{z_{41}}z_{42}) \\
+ &\left(\frac{1}{|z_{31}|^2} - \frac{1}{|z_{41}|^2} + \frac{1}{|z_{43}|^2}\right)\mathrm{Im}(z_{31}\overline{z_{41}}z_{43}) \\
+ &\left(\frac{1}{|z_{32}|^2} - \frac{1}{|z_{42}|^2} + \frac{1}{|z_{43}|^2}\right)\mathrm{Im}(z_{32}\overline{z_{42}}z_{43})
\end{aligned}$$

Moreover, if p has trace k with $1 \le k \le 3$, then for example, the term

$$z_{21}(\overline{z_{31}}z_{32}) = \frac{|z_{31}|^2|z_{32}|^2 + \overline{z_{31}}z_{32}z_{41}\overline{z_{42}}}{2 - k \pm \sqrt{\frac{1}{4} - |z_{31}|^2 - |z_{32}|^2 - |z_{43}|^2} \pm \sqrt{\frac{1}{4} - |z_{41}|^2 - |z_{42}|^2 - |z_{43}|^2}}$$

by Lemma 4.5*, which may not be real.*

Proof. It follows from the condition (O_4) that

$$\begin{aligned}
&(a_{11} + a_{22}) + \frac{\overline{z_{31}}z_{32}}{\overline{z_{21}}} + \frac{\overline{z_{41}}z_{42}}{\overline{z_{21}}} = 1, \quad (a_{11} + a_{33}) + \frac{\overline{z_{21}}\,\overline{z_{32}}}{\overline{z_{31}}} + \frac{\overline{z_{41}}z_{43}}{\overline{z_{31}}} = 1, \\
&(a_{22} + a_{33}) + \frac{z_{21}\overline{z_{31}}}{\overline{z_{32}}} + \frac{\overline{z_{42}}z_{43}}{\overline{z_{32}}} = 1, \quad (a_{11} + a_{44}) + \frac{\overline{z_{21}}\,\overline{z_{42}}}{\overline{z_{41}}} + \frac{\overline{z_{31}}\,\overline{z_{43}}}{\overline{z_{41}}} = 1, \\
&(a_{22} + a_{44}) + \frac{z_{21}\overline{z_{41}}}{\overline{z_{42}}} + \frac{\overline{z_{32}}\,\overline{z_{43}}}{\overline{z_{42}}} = 1, \quad (a_{33} + a_{44}) + \frac{z_{31}\overline{z_{41}}}{\overline{z_{43}}} + \frac{z_{32}\overline{z_{42}}}{\overline{z_{43}}} = 1.
\end{aligned}$$

Hence, adding both sides of these equations implies the first equality for the trace $\mathrm{tr}(p) = \sum_{j=1}^{4} a_{jj}$ in the statement. By adding and subtracting their complex conjugates on both sides we obtain the second and third equalities in the statement. □

Example 4.7. In particular, suppose that all z_{ij} for $4 \geq i > j \geq 1$ are equal to $z \in \mathbb{C} \setminus \{0\}$. It then follows from (T_4) that for $1 \leq k \leq 3$,

$$k = 2 - \frac{8}{3}z - \frac{4}{3}\overline{z}.$$

Thus, $2k = 4 - 8\mathrm{Re}(z)$ and $\mathrm{Im}(z) = 0$. Hence $z = \frac{2-k}{4}$.

If $k = 1$, then

$$a_{jj}^{\pm} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - 3\frac{1}{4^2}} = \frac{3}{4}, \frac{1}{4}$$

respectively for $1 \leq j \leq 4$. Therefore,

$$p = \begin{pmatrix} a_{11}^{-} = \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & a_{22}^{-} = \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & a_{33}^{-} = \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & a_{44}^{-} = \frac{1}{4} \end{pmatrix}.$$

If $k = 2$, then $z = 0$. This is a contradiction to $z \neq 0$. Hence $k \neq 2$.

If $k = 3$, then $a_{jj}^{\pm} = \frac{3}{4}, \frac{1}{4}$ respectively for $1 \leq j \leq 4$. Therefore,

$$p = \begin{pmatrix} a_{11}^{+} = \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & a_{22}^{+} = \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & a_{33}^{+} = \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & a_{44}^{+} = \frac{3}{4} \end{pmatrix}.$$

Example 4.8. In particular, suppose that all $|z_{ij}|$ for $4 \geq i > j \geq 1$ are equal to $r > 0$. Then for $1 \leq j \leq 4$,

$$a_{jj}^{\pm} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - 3r^2}, \quad \text{with } 0 < r^2 \leq \frac{1}{12}.$$

Suppose that the diagonal components of p are a_{jj}^{-} for $1 \leq j \leq 4$. Then

$$\mathrm{tr}(p) = 2 - 4\sqrt{\frac{1}{4} - 3r^2}.$$

If $\mathrm{tr}(p) = 1$, then $r = \frac{1}{4}$. Thus $a_{jj}^{-} = \frac{1}{4}$.

If $\mathrm{tr}(p) = 2$, then $r^2 = \frac{1}{12}$. Thus each $a_{jj} = a_{jj}^{-} = \frac{1}{2}$.

If $\mathrm{tr}(p) = 3$, then a contradiction is deduced.

Next suppose that the diagonal components of p are a_{jj}^{+} for $1 \leq j \leq 4$. Then

$$\mathrm{tr}(p) = 2 + 4\sqrt{\frac{1}{4} - 3r^2}.$$

If $\mathrm{tr}(p) = 1$, then a contradiction is deduced.
If $\mathrm{tr}(p) = 2$, then $r^2 = \frac{1}{12}$. Thus each $a^+_{jj} = \frac{1}{2}$.
If $\mathrm{tr}(p) = 3$, then $r = \frac{1}{4}$. Thus $a^+_{jj} = \frac{3}{4}$.

Table 3: The involved signs and only or no points of the diagonal components $a^{\pm}_{jj}$ $(1 \leq j \leq 4)$ with the off diagonal conditions limited

Trace	Sign	Diagonal	Off diagonal
0	$(-,-.-,-)$	Point $(0,0,0,0)$	$z_{ij} = 0$ $(1 \leq j < i \leq 4)$
1	$(-,-.-,-)$	Point $(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$	$z_{ij} = z \neq 0,$
	$(-,-.-,-)$	Point $(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$	$\lvert z_{ij}\rvert = r > 0$
	?	?	$(z_{ij} \neq z_{kl}, z_{ij} \neq 0)$
2		No point	$z_{ij} = z \neq 0$
	$(-,-,-,-)$	Point $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$\lvert z_{ij}\rvert = r > 0$
	$(+,+,+,+)$	Point $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$\lvert z_{ij}\rvert = r > 0$
	?	?	$(z_{ij} \neq z_{kl}, z_{ij} \neq 0)$
3	$(+,+.+,+)$	Point $(\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4})$	$z_{ij} = z \neq 0$
	$(+,+.+,+)$	Point $(\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4})$	$\lvert z_{ij}\rvert = r > 0$
	?	?	$(z_{ij} \neq z_{kl}, z_{ij} \neq 0)$
4	$(+,+.+,+)$	Point $(1,1,1,1)$	$z_{ij} = 0$

Remark. Note that the full degenerate case and some special generically non-degenerate cases are contained in the table above, but not are the partially degenerate cases. The more generic cases as round bracketed as ($z_{ij} \neq z_{kl}$ for some (i,j) and (k,l), $z_{ij} \neq 0$ for any (i,j)) with questions could be considered (partially) as in the Table 2, obtained in the previous section. But the details are not included by the deadline. In the generic cases, what is left to be determined is the possible signs and the corresponding limits with respect to off diagonal components.

Example 4.9. For instance to the last minute, suppose that for $i = 1, 2$ and $4 \geq j > i$, $z_{ji} = z \neq z_{43} = w$ with $zw \neq 0$. Namely, $p = (p_{ij}) =$

$$\begin{pmatrix} \frac{1}{2} \pm \sqrt{\frac{1}{4} - 3|z|^2} & \overline{z} & \overline{z} & \overline{z} \\ z & \frac{1}{2} \pm \sqrt{\frac{1}{4} - 3|z|^2} & \overline{z} & \overline{z} \\ z & z & \frac{1}{2} \pm \sqrt{\frac{1}{4} - 2|z|^2 - |w|^2} & \overline{w} \\ z & z & w & \frac{1}{2} \pm \sqrt{\frac{1}{4} - 2|z|^2 - |w|^2} \end{pmatrix}.$$

Then

$$\mathrm{tr}(p) = 2 - 2\mathrm{Re}(z) - \frac{2}{3}(2 + \frac{|z|^2}{|w|^2})\mathrm{Re}(w)$$

and

$$0 = 2\mathrm{Im}(z) + 2\frac{|z|^2}{|w|^2}\mathrm{Im}(w)$$

by using (T_4) of Proposition 4.6 above. Then

$$\frac{|z|^2}{\mathrm{Re}(w)^2 + \mathrm{Im}(w)^2} = \frac{|z|^2}{|w|^2} = -\frac{\mathrm{Im}(z)}{\mathrm{Im}(w)} > 0.$$

This is converted to the quadratic equation with respect to the imaginary part $\mathrm{Im}(w)$ as

$$\mathrm{Im}(w)^2 - \frac{|z|^2}{\mathrm{Im}(z)}\mathrm{Im}(w) + \mathrm{Re}(w)^2 = 0.$$

Hence, its real solutions are given by the solution formula as

$$\mathrm{Im}(w) = \frac{1}{2}\left(\frac{|z|^2}{\mathrm{Im}(z)} \pm \sqrt{\frac{|z|^4}{\mathrm{Im}(z)^2} - 4\mathrm{Re}(w)^2}\right)$$

with $2|\mathrm{Re}(w)| \leq \frac{|z|^2}{|\mathrm{Im}(z)|}$ assumed. Therefore, we obtain that

$$(\mathrm{tr}(p) - 2 + 2\mathrm{Re}(z))\mathrm{Im}(w) = -\frac{2}{3}(2\mathrm{Im}(w) - \mathrm{Im}(z))\mathrm{Re}(w).$$

Inserting to this equation $\mathrm{Im}(w)$ solved as real solutions we obtain a certain equation with respect to $\mathrm{Re}(w)$ with coefficients gives in terms of z and $\mathrm{tr}(p)$. Namely, converted is the following equation:

$$\begin{aligned}&\left(\mathrm{tr}(p) - 2 + 2\mathrm{Re}(z) + \frac{4}{3}\mathrm{Re}(w)\right)^2\left(\frac{|z|^4}{\mathrm{Im}(z)^2} - 4\mathrm{Re}(w)^2\right)\\ &= \left\{\frac{4}{3}\left(\mathrm{Im}(z) - \frac{|z|^2}{\mathrm{Im}(z)}\right)\mathrm{Re}(w) - (\mathrm{tr}(p) - 2 + 2\mathrm{Re}(z))\frac{|z|^2}{\mathrm{Im}(z)}\right\}^2.\end{aligned}$$

This equation with degree 4 may or not be solved by calculation by hand. If solved, then $w = \mathrm{Re}(w) + i\mathrm{Im}(w)$ is written in terms of $z = \mathrm{Re}(z) + i\mathrm{Im}(z)$ and $\mathrm{tr}(p)$.

The certainly possible admissible involved signs of the diagonal components $a_{jj}^{\pm}$ are given as

$$(+,-,+,-), \quad (+,-,-,+) \quad (-,+,+,-), \quad (-,+,-,+),$$

for which all the traces are the same 2. It may be shown that the other involved signs are not allowed in this case, likely.

The other similar cases with z and w exchanged among coordinates z_{ij} could be considered similarly as above.

5 The general matrix case

Solving the equation for projections of $M_n(\mathbb{C})$ implies that

Lemma 5.1. *If $p = (p_{ij})$ is a projection of $M_n(\mathbb{C})$, with $p = p^* = p^2$, then*

$$p = p(z_{ji}, n \geq j > i \geq 1) = \begin{pmatrix} a_{11} & \overline{z_{21}} & \cdots & \overline{z_{n1}} \\ z_{21} & a_{22} & \cdots & \overline{z_{n2}} \\ \vdots & \ddots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & a_{nn} \end{pmatrix}$$

with $a_{jj} \in \mathbb{R}(1 \leq j \leq n)$, $z_{ji} \in \mathbb{C}(1 \leq i < j \leq n)$, where

$$a_{11}^2 + \sum_{k=2}^{n} |z_{k1}|^2 = a_{11},$$
$$|z_{21}|^2 + a_{22}^2 + \sum_{k=3}^{n} |z_{k2}|^2 = a_{22}, \quad \cdots,$$
$$\sum_{k=1}^{l-1} |z_{lk}|^2 + a_{ll}^2 + \sum_{k=l+1}^{n} |z_{kl}|^2 = a_{ll}, \quad \cdots,$$
$$\sum_{k=1}^{n-1} |z_{nk}|^2 + a_{nn}^2 = a_{nn},$$

as the diagonal condition (D_n), and

$$(a_{11} + a_{22})\overline{z_{21}} + \sum_{k=3}^{n} \overline{z_{k1}} z_{k2} = \overline{z_{21}},$$
$$(a_{11} + a_{33})\overline{z_{31}} + \overline{z_{21} z_{32}} + \sum_{k=4}^{n} \overline{z_{k1}} z_{k3} = \overline{z_{31}},$$
$$z_{21}\overline{z_{31}} + (a_{22} + a_{33})\overline{z_{32}} + \sum_{k=4}^{n} \overline{z_{k2}} z_{k3} = \overline{z_{32}}, \quad \cdots,$$
$$(a_{11} + a_{nn})\overline{z_{n1}} + \sum_{k=2}^{n-1} \overline{z_{k1} z_{nk}} = \overline{z_{n1}}, \quad \cdots,$$
$$\sum_{k=1}^{n-2} z_{n-1,k}\overline{z_{n,k}} + (a_{n-1,n-1} + a_{nn})\overline{z_{n,n-1}} = \overline{z_{n,n-1}},$$

as the odd diagonal condition (O_n), so that each of the diagonal components a_{jj}

is solved from (D_n) *as*

$$a_{11} = a_{11}^{\pm}(z_{21}, \cdots, z_{n1}) \equiv \begin{cases} \frac{1}{2} \pm \sqrt{\frac{1}{4} - \sum_{k=2}^{n} |z_{k1}|^2} \in \mathbb{R} & \text{if } 0 \le \sum_{k=2}^{n} |z_{k1}|^2 \le \frac{1}{4}, \\ \frac{1}{2} \pm i\sqrt{\sum_{k=2}^{n} |z_{k1}|^2 - \frac{1}{4}} \notin \mathbb{R} & \text{if } \sum_{k=2}^{n} |z_{k1}|^2 > \frac{1}{4}, \end{cases},$$

$$a_{22} = a_{22}^{\pm}(z_{21}, z_{32}, \cdots, z_{n2}) \equiv$$

$$\frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_{21}|^2 - \sum_{k=3}^{n} |z_{k2}|^2} \in \mathbb{R} \quad \text{if } 0 \le |z_{21}|^2 + \sum_{k=3}^{n} |z_{k2}|^2 \le \frac{1}{4}, \quad \cdots,$$

$$a_{nn} = a_{nn}^{\pm}(z_{n1}, \cdots, z_{n,n-1}) \equiv \begin{cases} \frac{1}{2} \pm \sqrt{\frac{1}{4} - \sum_{k=1}^{n-1} |z_{nk}|^2} \in \mathbb{R} & \text{if } 0 \le \sum_{k=1}^{n-1} |z_{nk}|^2 \le \frac{1}{4}, \\ \frac{1}{2} \pm i\sqrt{\sum_{k=1}^{n-1} |z_{nk}|^2 - \frac{1}{4}} \notin \mathbb{R} & \text{if } \sum_{k=1}^{n-1} |z_{nk}|^2 > \frac{1}{4}. \end{cases}$$

Moreover, as the equality conditon (E_n), $a_{11}^+ = a_{11}^-$ *if and only if* $\sum_{k=2}^{n} |z_{k1}|^2 = \frac{1}{4}$, *and* $\cdots$, *and* $a_{nn}^+ = a_{nn}^-$ *if and only if* $\sum_{k=1}^{n-1} |z_{nk}|^2 = \frac{1}{4}$.

Corollary 5.2. *It follows from Lemma* 5.1 *the following*:

(1) *If all* $z_{ij} = 0$, *then* $a_{jj} = 0$ *or* 1 *for* $1 \le j \le n$.

(2) *If only one* $z_{ij} \ne 0$ *and the other all* $z_{kl} = 0$, *then these correspond to the* 2×2 *case or the split* 2×2 *case.*

(3) *Moreover, there are certainly the cases where there are distinct block-wise many of the* 2×2 *cases or the split* 2×2 *cases, as shown in the* 3×3 *or* 4×4 *cases.*

(4) *There is the rest of the other cases.*

We may say that a projection $p \in M_n(\mathbb{C})$ is **full degenerate** if it is in the case (1), and it is **partially degenerate** or a **generalized Bott** projection if in the cases (2) or (3), and it is **partially degenerate** and **not** a generalized Bott projection if in the case (4) but there is a zero component in the off diagonal part, and **generically non-degenerate** if in the case (4) and there are no zero components in the off diagonal part.

For convenience, we mainly consider the generically non-degenerate projections in what follows. The case of partially degenerate projections except generalized Bott projections may be considered elsewhere in the future.

Lemma 5.3. *As the bounded condition* (B_n), *in Lemma* 5.1, *if* $z_{21}, z_{31}, \cdots, z_{n1} \in \mathbb{C}$ *are fixed with* $0 \le \sum_{k=2}^{n} |z_{k1}|^2 \le \frac{1}{4}$, *then*

$$\sum_{k=3}^{n} |z_{k2}|^2 \le \frac{1}{4} - |z_{21}|^2, \quad \cdots\cdots, \quad \sum_{k=2}^{n-1} |z_{nk}|^2 \le \frac{1}{4} - |z_{n1}|^2,$$

and moreover, if $z_{32}, \cdots, z_{n2}$ *are fixed with* $\sum_{k=3}^{n} |z_{k2}|^2 \le \frac{1}{4} - |z_{21}|^2$, *then*

$$\cdots\cdots, \quad \sum_{k=3}^{n-1} |z_{nk}|^2 \le \frac{1}{4} - |z_{n1}|^2 - |z_{n2}|^2,$$

and this process continues finitely and inductively to end up to the last estimate of $|z_{n,n-1}|^2$ *as that for* $|z_{43}|^2$ *in the* 4×4 *case.*

Proposition 5.4. *Each connected component of the space* $P(M_n(\mathbb{C}))^{\sim}$ *is contained in the path-connected bounded set* $S^{2(n-1),2(n-3),\cdots,2}$ *defined by the diagonal condition* (D_n) *as the bounded condition* (B_n)*, which is homeomorphic to a fiber space over* $S^{2(n-1)}(\frac{1}{2})$ *with fibers obtained as the fiber spaces* $S^{2(n-2),\cdots,2}$ *over* $S^{2(n-2)}(\sqrt{\frac{1}{4}-r_1^2})$ *with fibers, similarly and inductively obtained as* $\cdots$. *Namely,*

$$\begin{array}{ccccc} S^{2(n-1),2(n-2),\cdots,2} & \longleftarrow & S^{2(n-2),\cdots,2} & \longleftarrow \cdots \longleftarrow & S^2(\sqrt{\frac{1}{4}-\sum_{k=1}^{n-2} r_k^2}) \\ \downarrow & & \downarrow & \downarrow & \\ S^{2(n-1)}(\frac{1}{2}) & & S^{2(n-2)}(\sqrt{\frac{1}{4}-r_1^2}) & \cdots & \end{array}$$

where the positive real parameters $r_1, \cdots, r_{n-2}$ *vary continuously, with* $0 \leq r_2^2 \leq \frac{1}{4}, \cdots, 0 \leq \sum_{k=1}^{n-2} r_k^2 \leq \frac{1}{4}$.

Lemma 5.5. *Suppose that* p *is a generically non-degenerate projection of* $M_n(\mathbb{C})$.

If p *has trace* k *with* $1 \leq k \leq n-1$*, then for example,*

$$\overline{z_{21}} = \frac{\sum_{k=3}^{n} \overline{z_{k1}} z_{k2}}{1 - k + \sum_{k=3}^{n} a_{kk}^{\pm}},$$

Proof. The condition (O_n) implies that

$$\sum_{k=3}^{n} \overline{z_{k1}} z_{k2} = (1 - a_{11} - a_{22})\overline{z_{21}} = (1 - \mathrm{tr}(p) + \sum_{k=3}^{n} a_{kk})\overline{z_{21}},$$

so that the condition (D_n) implies the formula in the statement. □

Proposition 5.6. *If* p *is a generically non-degenerate projection of* $M_n(\mathbb{C})$*, then the trace of* p *is given by, as the trace condition* (T_n)*,*

$$\begin{aligned} \mathrm{tr}(p) = \sum_{j=1}^{n} a_{jj} &= \frac{n}{2} - \frac{1}{n-1}\frac{1}{|z_{21}|^2}\sum_{k=3}^{n} z_{21}\overline{z_{k1}}z_{k2} \\ &- \frac{1}{n-1}\frac{1}{|z_{31}|^2}\left(\overline{z_{21}}z_{31}\overline{z_{32}} + \sum_{k=4}^{n} z_{31}\overline{z_{k1}}z_{k3}\right) \\ &- \frac{1}{n-1}\frac{1}{|z_{32}|^2}\left(z_{21}\overline{z_{31}}z_{32} + \sum_{k=4}^{n} z_{32}\overline{z_{k2}}z_{k3}\right) - \cdots \\ &- \frac{1}{n-1}\frac{1}{|z_{n1}|^2}\sum_{k=2}^{n-1} \overline{z_{k1}}z_{n1}\overline{z_{nk}} - \cdots \\ &- \frac{1}{n-1}\frac{1}{|z_{n,n-1}|^2}\sum_{k=1}^{n-2} z_{n-1,k}\overline{z_{n,k}}z_{n,n-1}. \end{aligned}$$

Proof. The condition (O_n) is converted as

$$a_{11}+a_{22}=1-\frac{1}{\overline{z_{21}}}\sum_{k=3}^{n}\overline{z_{k1}}z_{k2},\quad a_{11}+a_{33}=1-\frac{1}{\overline{z_{31}}}\left(\overline{z_{21}z_{32}}+\sum_{k=4}^{n}\overline{z_{k1}}z_{k3}\right),$$

$$a_{22}+a_{33}=1-\frac{1}{\overline{z_{32}}}\left(z_{21}\overline{z_{31}}+\sum_{k=4}^{n}\overline{z_{k2}}z_{k3}\right),\quad \cdots,$$

$$\cdots,\quad a_{11}+a_{nn}=1-\frac{1}{\overline{z_{n1}}}\sum_{k=2}^{n-1}\overline{z_{k1}z_{nk}},\quad \cdots,$$

$$\cdots,\quad a_{n-1,n-1}+a_{nn}=1-\frac{1}{\overline{z_{n,n-1}}}\sum_{k=1}^{n-2}z_{n-1,k}\overline{z_{n,k}}.$$

Adding both sides of these $\frac{(n-1)n}{2}$ equations with 2 diagonal components implies the trace of p as in the statement. □

Example 5.7. In particular, suppose that all z_{ij} for $1\leq j<i\leq n$ are equal to $z\in\mathbb{C}\setminus\{0\}$. It then follows from the condition (T_n) that for $1\leq k\leq n-1$,

$$\begin{aligned}k=&\frac{n}{2}-\frac{1}{n-1}\{\frac{(n-2)(n-1)}{2}+\frac{(n-2)(n-1)}{2}+\cdots\\&+(n-2+n-3)+(n-2)\}z\\&-\frac{1}{n-1}\{\frac{(n-2)(n-1)}{2}+\frac{(n-3)(n-2)}{2}+\cdots+1+0\}\overline{z}\\\equiv&\frac{n}{2}-c_nz-d_n\overline{z}.\end{aligned}$$

By adding and subtracting the complex conjugates of both sides, we obtain

$$\begin{aligned}2k&=n-2(c_n+d_n)\mathrm{Re}(z)=n-n(n-2)\mathrm{Re}(z),\\0&=0-2i(c_n-d_n)\mathrm{Im}(z),\end{aligned}$$

with $c_n>d_n$. Hence $z\in\mathbb{R}$, so that

$$z=\frac{n-2k}{n(n-2)}.$$

It then follows that

$$\begin{aligned}a_{jj}^{\pm}&=\frac{1}{2}\pm\sqrt{\frac{1}{4}-(n-1)\frac{(n-2k)^2}{n^2(n-2)^2}}\quad(1\leq j\leq n)\\&=\frac{1}{2}\pm\frac{\sqrt{n^2(n-2)^2-4(n-1)(n-2k)^2}}{2n(n-2)}.\end{aligned}$$

In particular, if $k=1$, then $z=\frac{1}{n}$ and

$$a_{jj}^{\pm}=\frac{1}{2}\pm\sqrt{\frac{1}{4}-\frac{n-1}{n^2}}=\frac{1}{2}\pm\left(\frac{n-2}{2n}\right)=1-\frac{1}{n},\frac{1}{n}.$$

Therefore, we get

$$p = \begin{pmatrix} a_{11}^- = \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \ddots & \vdots \\ \frac{1}{n} & \cdots & a_{nn}^- = \frac{1}{n} \end{pmatrix}.$$

If $k = n-1$, then $z = -\frac{1}{n}$ and $a_{jj}^{\pm} = 1 - \frac{1}{n}, \frac{1}{n}$. Therefore, we get

$$p = \begin{pmatrix} a_{11}^+ = 1 - \frac{1}{n} & \frac{-1}{n} & \cdots & \frac{-1}{n} \\ \frac{-1}{n} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{-1}{n} \\ \frac{-1}{n} & \cdots & \frac{-1}{n} & a_{nn}^+ = 1 - \frac{1}{n} \end{pmatrix}.$$

If n is even as $n = 2m$, then p does not have trace m. If p has trace m, then $z = 0$, a contradiction.

At this moment, we could not check whether the other cases happen or not.

Example 5.8. In particular, suppose that all $|z_{ij}|$ for $n \geq i > j \geq 1$ are equal to $r > 0$. Then for $1 \leq j \leq n$,

$$a_{jj}^{\pm} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - (n-1)r^2}, \quad \text{with } 0 < r^2 \leq \frac{1}{4(n-1)}.$$

Suppose that the diagonal components of p are a_{jj}^- for $1 \leq j \leq n$. Then

$$\mathrm{tr}(p) = \frac{n}{2} - n\sqrt{\frac{1}{4} - (n-1)r^2}.$$

In particular, if $\mathrm{tr}(p) = 1$, then $r = \frac{1}{n}$ and each $a_{jj}^- = \frac{1}{n}$. Suppose that each $z_{ij} = re^{i\theta_{ij}}$ with $\theta_{ij} \in \mathbb{R}$. Then

$$p = \begin{pmatrix} a_{11}^- = \frac{1}{n} & \cdots & \frac{1}{n}e^{-i\theta_{n1}} \\ \vdots & \ddots & \vdots \\ \frac{1}{n}e^{i\theta_{n1}} & \cdots & a_{nn}^- = \frac{1}{n} \end{pmatrix}$$

with a suitable angle condition on θ_{ij}.

If $\mathrm{tr}(p) = \frac{n}{2}$ with n even, then $r^2 = \frac{1}{4(n-1)}$. Thus each $a_{jj}^- = \frac{1}{2}$.

If $\mathrm{tr}(p) > \frac{n}{2}$, then a contradiction on signs is deduced.

Next suppose that the diagonal components of p are a_{jj}^+ for $1 \leq j \leq n$. Then

$$\mathrm{tr}(p) = \frac{n}{2} + n\sqrt{\frac{1}{4} - (n-1)r^2}.$$

If $\mathrm{tr}(p) < \frac{n}{2}$, then a contradiction on signs is deduced.

If $\mathrm{tr}(p) = \frac{n}{2}$ with n even, then $r^2 = \frac{1}{4(n-1)}$. Thus each $a_{jj}^+ = \frac{1}{2}$.

If $\mathrm{tr}(p) = n - 1$, then $r = \frac{1}{n}$ and each $a_{jj}^{+} = 1 - \frac{1}{n}$. Then

$$p = \begin{pmatrix} a_{11}^{+} = 1 - \frac{1}{n} & \cdots & \frac{1}{n}e^{-i\theta_{n1}} \\ \vdots & \ddots & \vdots \\ \frac{1}{n}e^{i\theta_{n1}} & \cdots & a_{nn}^{+} = 1 - \frac{1}{n} \end{pmatrix}$$

with a suitable angle condition on θ_{ij}.

Table 4: The involved signs and only or no points of the diagonal components $a_{jj}^{\pm}$ $(1 \leq j \leq n)$ with the off diagonal conditions limited

Trace	Sign	Diagonal	Off diagonal
0	All minus	Point $0_n = (0, \cdots, 0)$	$z_{ij} = 0$ $(1 \leq j < i \leq n)$
1	$(-, \cdots, -)$ $(-, \cdots, -)$?	Point $(\frac{1}{n}, \cdots, \frac{1}{n})$ Point $(\frac{1}{n}, \cdots, \frac{1}{n})$?	$z_{ij} = z \neq 0, = \frac{1}{n}$ $\lvert z_{ij}\rvert = r > 0, = \frac{1}{n}$ $(z_{ij} \neq z_{kl}, z_{ij} \neq 0)$
$< \frac{n}{2}$	$(+, \cdots, +)$?	No point ?	$\lvert z_{ij}\rvert = r > 0$ $(z_{ij} \neq z_{kl}, z_{ij} \neq 0)$
m $(n = 2m)$	 $(-, \cdots, -)$ $(+, \cdots, +)$?	No point Point $(\frac{1}{2}, \cdots, \frac{1}{2})$ Point $(\frac{1}{2}, \cdots, \frac{1}{2})$?	$z_{ij} = z \neq 0$ $\lvert z_{ij}\rvert = r > 0, = \frac{1}{2\sqrt{n-1}}$ $\lvert z_{ij}\rvert = r > 0, = \frac{1}{2\sqrt{n-1}}$ $(z_{ij} \neq z_{kl}, z_{ij} \neq 0)$
$> \frac{n}{2}$	$(-, \cdots, -)$?	No point ?	$\lvert z_{ij}\rvert = r > 0$ $(z_{ij} \neq z_{kl}, z_{ij} \neq 0)$
$n - 1$	$(+, \cdots, +)$ $(+, \cdots, +)$?	Point $(1 - \frac{1}{n}, \cdots, 1 - \frac{1}{n})$ Point $(1 - \frac{1}{n}, \cdots, 1 - \frac{1}{n})$?	$z_{ij} = z \neq 0, = \frac{-1}{n}$ $\lvert z_{ij}\rvert = r > 0, = \frac{-1}{n}$ $(z_{ij} \neq z_{kl}, z_{ij} \neq 0)$
n	All plus	Point $1_n = (1, \cdots, 1)$	$z_{ij} = 0$

Remark. Note that the full degenerate case and some special generically non-degenerate cases are contained in the table above, but not are the partially degenerate cases. The more generic cases as round bracketed as ($z_{ij} \neq z_{kl}$ for some (i, j) and (k, l), $z_{ij} \neq 0$ for any (i, j)) with questions could be considered (partially) as in the Table 2 or in Example 4.9. But the details are not included by the deadline. In the generic cases, what is left to be determined is the possible signs and the corresponding limits with respect to off diagonal components, but after solving the corresponding algebraic equations with some higher degrees.

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