



Title	q-DIFFERENCE EQUATIONS FOR q-HYPERGEOMETRIC INTEGRALS OF TYPE G_2
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## $q$ -DIFFERENCE EQUATIONS FOR $q$ -HYPERGEOMETRIC INTEGRALS OF TYPE $G_2$

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ABSTRACT. We provide a simpler proof for an infinite product expression of Gustafson's  $q$ -beta integral of type  $G_2$  with 4 parameters. We extend Gustafson's  $q$ -integral to a  $q$ -hypergeometric integral of type  $G_2$  with 6 parameters. Under a constraint of the parameters called the balancing condition, we obtain two explicit forms of  $q$ -difference equations satisfied by the  $q$ -hypergeometric integral of type  $G_2$ . Taking limit for a parameter, the  $q$ -hypergeometric integral of type  $G_2$  degenerates to Gustafson's  $q$ -integral, and one of two  $q$ -difference equations becomes that satisfied by Gustafson's  $q$ -integral. Using this we consequently have an alternative proof for the infinite product of Gustafson's  $q$ -beta integral again.

### 1. INTRODUCTION

The beta integral

$$(1.1) \quad B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx \quad (\operatorname{Re} \alpha > 0, \operatorname{Re} \beta > 0)$$

satisfies the formula

$$(1.2) \quad B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)},$$

where  $\Gamma(\alpha)$  is the gamma function given as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \quad (\operatorname{Re} \alpha > 0).$$

The formula (1.2) has a great significance in classical analysis and is applied variously not only limited to Mathematics. For instance, the orthogonal polynomials associated with the integrand of (1.1) as weight functions are called the Jacobi polynomials, and their properties are studied precisely from the view of their theories and applications. As a generalization of (1.2), the Selberg integral as a multivariable beta integral can be written as the product of gamma functions, i.e.,

$$\begin{aligned} & \frac{1}{n!} \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^{\alpha-1} (1-x_i)^{\beta-1} \prod_{1 \leq j < k \leq n} |x_j - x_k|^{2\tau} dx_1 dx_2 \cdots dx_n \\ &= \prod_{j=1}^n \frac{\Gamma(\alpha + (j-1)\tau)\Gamma(\beta + (j-1)\tau)\Gamma(j\tau)}{\Gamma(\alpha + \beta + (n+j-2)\tau)\Gamma(\tau)}, \end{aligned}$$

where  $\operatorname{Re} \alpha > 0$ ,  $\operatorname{Re} \beta > 0$  and  $\operatorname{Re} \tau > -\min\{1/n, \operatorname{Re} \alpha/(n-1), \operatorname{Re} \beta/(n-1)\}$ . This formula coincides with (1.2) if  $n = 1$ , and is also fundamental to the theory of

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multivariable orthogonal polynomials. (See the recent reference [3] for topics relevant to the Selberg integral.)

Using Jackson integral  $\int_0^1 f(x) d_q x = (1-q) \sum_{i=0}^{\infty} f(q^i) q^i$ ,  $q$ -analogues of the beta integral and the gamma function are given as

$$B_q(\alpha, \beta) = \int_0^1 x^\alpha \frac{(qx; q)_\infty}{(q^\beta x; q)_\infty} \frac{d_q x}{x}, \quad \Gamma_q(\alpha) = \frac{(q; q)_\infty}{(q^\alpha; q)_\infty} (1-q)^{1-\alpha},$$

where  $|q| < 1$ . Here we used the symbol  $(u; q)_\infty := \prod_{i=0}^{\infty} (1 - q^i u)$ . For simplicity we also use the symbol  $(u_1, u_2, \dots, u_n)_\infty := (u_1; q)_\infty (u_2; q)_\infty \cdots (u_n; q)_\infty$ . As  $q \rightarrow 1$ ,  $B_q(\alpha, \beta)$  and  $\Gamma_q(\alpha)$  become  $B(\alpha, \beta)$  and  $\Gamma(\alpha)$ , respectively. In 1980 Askey [1] established a  $q$ -analogue of the Selberg integral given as

$$(1.3) \quad \int_{z_1=0}^1 \int_{z_2=0}^{q^\tau z_1} \cdots \int_{z_n=0}^{q^\tau z_{n-1}} \prod_{i=1}^n z_i^\alpha \frac{(qz_i; q)_\infty}{(q^\beta z_i; q)_\infty} \\ \times \prod_{1 \leq j < k \leq n} z_j^{2\tau-1} \frac{(q^{1-\tau} z_k / z_j; q)_\infty}{(q^\tau z_k / z_j; q)_\infty} (z_j - z_k) \frac{d_q z_n}{z_n} \cdots \frac{d_q z_2}{z_2} \frac{d_q z_1}{z_1} \\ = q^{\alpha\tau} \binom{n}{2} + 2\tau^2 \binom{n}{3} \prod_{j=1}^n \frac{\Gamma_q(\alpha + (j-1)\tau) \Gamma_q(\beta + (j-1)\tau) \Gamma_q(j\tau)}{\Gamma_q(\alpha + \beta + (n+j-2)\tau) \Gamma_q(\tau)},$$

where  $\operatorname{Re} \alpha > 0$  and  $\operatorname{Re} \alpha + (n-1)\operatorname{Re} \tau > 0$ . (In [9] an explanation for the formula (1.3) supporting this paper is provided.) After (1.3) appeared, with a great deal of researches developed in the 1990s for the Macdonald orthogonal polynomials associated with root systems,  $q$ -beta integrals possessing Weyl group symmetry associated with root systems were studied. The most typical one associated with root systems is the complex integral given as

$$(1.4) \quad \frac{(q; q)_\infty}{2(2\pi\sqrt{-1})} \int_{|z|=1} \frac{(z^2, z^{-2}; q)_\infty}{\prod_{k=1}^4 (a_k z, a_k z^{-1}; q)_\infty} \frac{dz}{z} = \frac{(a_1 a_2 a_3 a_4; q)_\infty}{\prod_{1 \leq i < j \leq 4} (a_i a_j; q)_\infty},$$

where  $|a_k| < 1$  ( $k = 1, \dots, 4$ ). We call the left-hand side of (1.4) the Askey–Wilson integral (or  $q$ -beta integral of type  $BC_1$  [8]), which gives the orthogonal norm of the Askey–Wilson orthogonal polynomials [2]. Nassrallah and Rahman [13] established an extension of (1.4) given as

$$(1.5) \quad \frac{(q; q)_\infty}{2(2\pi\sqrt{-1})} \int_{|z|=1} \frac{(z^{\pm 2}, qa_6^{-1} z^{\pm 1}; q)_\infty}{\prod_{k=1}^5 (a_k z^{\pm 1}; q)_\infty} \frac{dz}{z} = \frac{\prod_{i=1}^5 (qa_6^{-1} a_i^{-1}; q)_\infty}{\prod_{1 \leq i < j \leq 5} (a_i a_j; q)_\infty}$$

under the balancing condition  $a_1 a_2 \cdots a_6 = q$ , where  $|a_k| < 1$  ( $k = 1, \dots, 5$ ). Here we used the symbols  $(u^{\pm 2}; q)_\infty = (u^2, u^{-2}; q)_\infty$  and  $(cu^{\pm 1}; q)_\infty = (cu, cu^{-1}; q)_\infty$ . By taking limit  $a_5 \rightarrow 0$  the Askey–Wilson integral (1.4) is obtained from the Nassrallah–Rahman integral (1.5) as a special case. Although the Nassrallah–Rahman integral does not give a norm for specific orthogonal polynomials, it has higher symmetry than the Askey–Wilson integral, and consequently the proof of the formula (1.5) become simpler than that of the formula (1.4). The formulas (1.4) and (1.5) can be extended to the multidimensional case ( $q$ -Selberg contour integral of type  $BC_n$ ), which gives the norm of the Macdonald–Koornwinder multivariable orthogonal polynomials, and topics around the integrals of type  $BC_n$  are still much actively researched area. We

note that in the last two decades, several elliptic extensions ( $p, q$ -analogue) of the  $q$ -beta integrals have been studied, especially for those of type  $BC_n$  by van Diejen and Spiridonov [17], Spiridonov [15], Rains [14] (see also [11] for explanation supporting this paper).

On the other hand, compared with the development of the  $q$ -beta integrals associated with classical root systems, research on those associated with the exceptional root systems seems to be not fully studied yet at present. For the root system  $G_2$  Gustafson [5, 6] showed the following.

**Proposition 1.1.** *Suppose that  $a_k \in \mathbb{C}^*$  ( $1 \leq k \leq 4$ ) satisfy  $|a_k| < 1$ . Then,*

$$(1.6) \quad \frac{(q; q)_\infty^2}{12(2\pi\sqrt{-1})^2} \iint_{\mathbb{T}^2} \frac{\prod_{1 \leq i < j \leq 3} (x_i x_j, x_i^{-1} x_j, x_i x_j^{-1}, x_i^{-1} x_j^{-1}; q)_\infty}{\prod_{i=1}^3 \prod_{k=1}^4 (a_k x_i, a_k x_i^{-1}; q)_\infty} \frac{dx_1}{x_1} \frac{dx_2}{x_2}$$

$$= \frac{(a_1^2 a_2^2 a_3^2 a_4^2; q)_\infty}{(a_1 a_2 a_3 a_4; q)_\infty} \prod_{i=1}^4 \frac{(a_i; q)_\infty}{(a_i^2; q)_\infty} \prod_{1 \leq i < j \leq 4} \frac{1}{(a_i a_j; q)_\infty} \prod_{1 \leq i < j < k \leq 4} \frac{1}{(a_i a_j a_k; q)_\infty},$$

where  $x_3 = x_1^{-1} x_2^{-1}$  and  $\mathbb{T}^2$  is the 2-dimensional torus given as

$$\mathbb{T}^2 = \{(x_1, x_2) \in (\mathbb{C})^2 \mid |x_i| = 1 \ (i = 1, 2)\}$$

In spite of its simple appearance no short proofs for the formula (1.6) are known. One of our aims is to give a simpler proof for (1.6). The other aim is to investigate a generalized integral of Nassrallah–Rahman type for the case of  $G_2$  defined as

$$(1.7) \quad \frac{(q; q)_\infty^2}{12(2\pi\sqrt{-1})^2} \iint_{\mathbb{T}^2} \frac{\prod_{1 \leq i < j \leq 3} (x_i^\pm x_j^\pm, q a_6^{-1} x_i^\pm x_j^\pm; q)_\infty}{\prod_{i=1}^3 \prod_{k=1}^5 (a_k x_i^\pm; q)_\infty} \frac{dx_1}{x_1} \frac{dx_2}{x_2},$$

under the conditions  $x_1 x_2 x_3 = 1$  and  $a_1 a_2 \cdots a_6 = -q$ . For simplicity here we use the symbol  $(cu^\pm v^\pm; q)_\infty = (cuv, cu^{-1}v, cuv^{-1}, cu^{-1}v^{-1}; q)_\infty$ . Gustafson's integral (the left-hand side of (1.6)) is included in (1.7) as the limiting case  $a_5 \rightarrow 0$ . Although the integral (1.7) no longer has product expression by gamma functions as the right-hand side of (1.6), it satisfies two independent  $q$ -difference equations of rank 2. We provide the explicit forms of these two equations (see Theorems 7.1 and 7.2). As a corollary of the theorem we can understand that, when  $a_5 \rightarrow 0$ , one of the  $q$ -difference equations degenerate to that of rank 1 satisfied by Gustafson's integral. This consequently gives another alternative proof for the formula (1.6).

This paper is organized as follows. After defining basic terminology in Section 2, we explain in Section 3 a way to derive the formula (1.4) for the Askey–Wilson integral ( $BC_1$  case), before proving the formula (1.6) for Gustafson's integral of type  $G_2$ . Since the arguments for both  $BC_1$  and  $G_2$  cases are completely parallel, this section for  $BC_1$  case would be instructive to understand the strategy of two steps for  $G_2$  case. The first step is to derive the  $q$ -difference equations (two-term recurrence relations) for the integral with respect to its parameters by using  $q$ -Stokes' theorem. Once we had these recurrence relations, using them repeatedly we see that the integral can be expressed as a product form up to a multiplicative constant. The next step is to determine the indefinite constant using a special value of the integral at some specific point. We enumerate several special values for the integral of type  $BC_1$  which are simply computed at the corresponding specific points. In Section 4 we explain the derivation of the formula (1.5) for the Nassrallah–Rahman integral ( $BC_1$

case with a balancing condition). The argument of this section is almost parallel to that of Section 3 except the treatment of the balancing condition. In Section 5 we recall terminology of the root system  $G_2$  and its Weyl group  $W$ . In addition to this we introduce two  $\mathbb{C}$ -vector subspaces  $\mathcal{F}_2$  and  $\mathcal{F}_4$  in the  $\mathbb{C}$ -vector space of  $W$ -invariant Laurent polynomials, and define the  $\mathbb{C}$ -bases of  $\mathcal{F}_i$ , ( $i = 2, 4$ ) that consist of Lagrange interpolation polynomials associated with some specific points. These bases are important and necessary to apply  $q$ -Stokes' theorem to the integrals of type  $G_2$  when we derive  $q$ -difference equations for the integrals of type  $G_2$  in the succeeding sections. Section 6 is devoted to a derivation for the formula (1.6) of Gustafson's integral of type  $G_2$ . This is one of our main results. Although calculation we need is more complex than the  $BC_1$  case, the strategy of two steps is still the same as the  $BC_1$  case. In Section 7 we define Nassrallah–Rahman type integral for  $G_2$  case (the integral of type  $G_2$  with a balancing condition). We compute the explicit forms of two  $q$ -difference equations which the Nassrallah–Rahman integral of type  $G_2$  satisfies (see Theorems 7.1 and 7.2). This is the other result of ours.

Lastly we note that the contents of this paper is fundamentally based on the thesis of the second author [16]. We remark that the elliptic version of Gustafson's integral (1.6) was recently proved in [10].

## 2. PRELIMINARIES

Throughout this paper we fix  $q \in \mathbb{C}^*$  with  $|q| < 1$ . We use the  $q$ -shifted factorials for  $x \in \mathbb{C}$  as

$$(x; q)_\infty := \prod_{i=0}^{\infty} (1 - q^i x)$$

and

$$(x; q)_n := \frac{(x; q)_\infty}{(q^n x; q)_\infty} = \begin{cases} (1-x)(1-qx)(1-q^2x) \cdots (1-q^{n-1}x) & \text{if } n = 1, 2, \dots, \\ 1 & \text{if } n = 0, \\ \frac{1}{(1-q^{-1}x)(1-q^{-2}x) \cdots (1-q^n x)} & \text{if } n = -1, -2, \dots \end{cases}$$

We also use the symbol

$$(x_1, x_2, \dots, x_m; q)_\infty := (x_1; q)_\infty (x_2; q)_\infty \cdots (x_m; q)_\infty.$$

By definition we have

$$(2.1) \quad (x^2; q)_\infty = (x, -x, q^{\frac{1}{2}}x, -q^{\frac{1}{2}}x; q)_\infty.$$

In particular, if  $x = q^{\frac{1}{2}}$ , then we have  $(q; q)_\infty = (q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q, -q; q)_\infty$ , so that

$$(2.2) \quad 1 = (q^{\frac{1}{2}}, -q^{\frac{1}{2}}, -q; q)_\infty.$$

For  $u, v \in \mathbb{C}^*$  we set

$$(2.3) \quad e(u, v) := u + u^{-1} - v - v^{-1} = u^{-1}(1 - uv)(1 - u/v),$$

which satisfies

$$e(u, v) = -e(v, u), \quad e(u, v) = e(u^{-1}, v), \quad e(u, v) + e(v, w) = e(u, w),$$

and

$$e(u, v)e(w, x) - e(u, w)e(v, x) + e(u, x)e(v, w) = 0.$$

For  $z \in \mathbb{C}^*$  we define  $\vartheta(z; q)$  by the bilateral series

$$(2.4) \quad \vartheta(z; q) := \sum_{\nu=-\infty}^{\infty} (-z)^\nu q^{\binom{\nu}{2}},$$

which converges uniformly on compact sets of  $\mathbb{C}^*$  and satisfies

$$(2.5) \quad \vartheta(z; q) = (z, qz^{-1}, q; q)_\infty.$$

The identity (2.5) is called Jacobi's triple product formula [4].

### 3. ASKEY–WILSON INTEGRAL

The aim of this section is to provide a way to prove the following identity.

**Proposition 3.1.** *Suppose that  $a_k \in \mathbb{C}^*$  ( $1 \leq k \leq 4$ ) satisfy  $|a_k| < 1$ . Then, we have*

$$(3.1) \quad \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{(z^2, z^{-2}; q)_\infty}{\prod_{k=1}^4 (a_k z, a_k z^{-1}; q)_\infty} \frac{dz}{z} = \frac{2}{(q; q)_\infty} \frac{(a_1 a_2 a_3 a_4; q)_\infty}{\prod_{1 \leq i < j \leq 4} (a_i a_j; q)_\infty},$$

where  $\mathbb{T}$  is the unit circle  $\{z \in \mathbb{C} \mid |z| = 1\}$  traversed along the positive direction.

The left-hand side of (3.1) is called the *Askey–Wilson integral* [2, 8]. Throughout this section, we define the function  $\Phi(z)$  on  $\mathbb{C}^*$  by

$$(3.2) \quad \Phi(z) := \frac{(z^2, z^{-2}; q)_\infty}{\prod_{k=1}^4 (a_k z, a_k z^{-1}; q)_\infty}$$

and also denote by  $I(a_1, a_2, a_3, a_4)$  the left-hand side of (3.1), i.e.,

$$(3.3) \quad I(a_1, a_2, a_3, a_4) := \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \Phi(z) \frac{dz}{z}.$$

On the other hand, we define  $P(a_1, a_2, a_3, a_4)$  by the infinite product

$$(3.4) \quad P(a_1, a_2, a_3, a_4) := \frac{(a_1 a_2 a_3 a_4; q)_\infty}{\prod_{1 \leq i < j \leq 4} (a_i a_j; q)_\infty},$$

so that the identity (3.1) is rewritten as

$$I(a_1, a_2, a_3, a_4) = \frac{2}{(q; q)_\infty} P(a_1, a_2, a_3, a_4)$$

for  $a_k \in \mathbb{C}^*$  ( $1 \leq k \leq 4$ ) satisfying  $|a_k| < 1$ . Before proving this identity, we state the  $q$ -Stokes' theorem in the following section.

**3.1.  $q$ -Difference operator  $\nabla_{q,z}$  and  $q$ -Stokes' theorem.** For an arbitrary function  $f(z)$  on  $\mathbb{C}^*$  we define the  $q$ -shift operator  $T_{q,z}$  by  $T_{q,z}f(z) = f(qz)$ .

**Lemma 3.2.** *Let  $F_+(z)$  and  $F_-(z)$  be functions specified by*

$$(3.5) \quad F_+(z) := \frac{1}{z} \frac{\prod_{k=1}^4 (1 - a_k z)}{1 - z^2} = -\frac{1}{z - z^{-1}} z^{-2} \prod_{k=1}^4 (1 - a_k z),$$

$$(3.6) \quad F_-(z) := F_+(z^{-1}) = z \frac{\prod_{k=1}^4 (1 - a_k z^{-1})}{1 - z^{-2}}.$$

Then, it follows that

$$(3.7) \quad \frac{T_{q,z}\Phi(z)}{\Phi(z)} = -\frac{F_+(z)}{T_{q,z}F_-(z)}.$$

**Proof.** By the definition (3.2) of  $\Phi(z)$  we have

$$\begin{aligned} \frac{\Phi(qz)}{\Phi(z)} &= \frac{(q^2 z^2, q^{-2} z^{-2}; q)_\infty}{(z^2, z^{-2}; q)_\infty} \prod_{k=1}^4 \frac{(a_k z, a_k z^{-1}; q)_\infty}{(q a_k z, q^{-1} a_k z^{-1}; q)_\infty} \\ &= \frac{(1 - q^{-2} z^{-2})(1 - q^{-1} z^{-2})}{(1 - z^2)(1 - qz^2)} \prod_{k=1}^4 \frac{1 - a_k z}{1 - q^{-1} a_k z^{-1}} \\ &= -q^{-1} z^{-2} \frac{1 - q^{-2} z^{-2}}{1 - z^2} \prod_{k=1}^4 \frac{1 - a_k z}{1 - q^{-1} a_k z^{-1}} = -\frac{F_+(z)}{F_-(qz)}, \end{aligned}$$

which completes the proof. □

For an arbitrary meromorphic function  $\varphi(z)$  on  $\mathbb{C}^*$  we define the symbol

$$(3.8) \quad \langle \varphi(z) \rangle := \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \varphi(z) \Phi(z) \frac{dz}{z}$$

In particular, from (3.3) we have  $\langle 1 \rangle = I(a_1, a_2, a_3, a_4)$ . The following is a technical key lemma of this paper.

**Proposition 3.3** ( $q$ -Stokes' theorem). *For an arbitrary meromorphic function  $\varphi(z)$  on  $\mathbb{C}^*$ , let  $\nabla_{q,z}$  be operator specified by*

$$(3.9) \quad (\nabla_{q,z}\varphi)(z) := F_-(z)\varphi(z) + F_+(z)T_{q,z}\varphi(z).$$

Suppose that  $|a_k| < 1$  ( $k = 1, 2, 3, 4$ ). For an arbitrary holomorphic function  $\varphi(z)$  on  $\mathbb{C}^*$ , it follows that

$$(3.10) \quad \langle \nabla_{q,z}\varphi(z) \rangle = 0.$$

**Proof.** By the definition of  $\nabla_{q,z}$  we have

$$\begin{aligned}
(3.11) \quad \langle \nabla_{q,z} \varphi(z) \rangle &= \int_{\mathbb{T}} \Phi(z) \nabla_{q,z} \varphi(z) \frac{dz}{z} = \int_{\mathbb{T}} \Phi(z) \left( F_-(z) \varphi(z) + F_+(z) \varphi(qz) \right) \frac{dz}{z} \\
&= \int_{\mathbb{T}} \Phi(z) \left( F_-(z) \varphi(z) - \frac{\Phi(qz)}{\Phi(z)} F_-(qz) \varphi(qz) \right) \frac{dz}{z} \quad (\text{from (3.7)}) \\
&= \int_{\mathbb{T}} \left( \Phi(z) F_-(z) \varphi(z) - \Phi(qz) F_-(qz) \varphi(qz) \right) \frac{dz}{z} \\
&= \int_{\mathbb{T}} \Phi(z) F_-(z) \varphi(z) \frac{dz}{z} - \int_{\mathbb{T}} \Phi(qz) F_-(qz) \varphi(qz) \frac{dz}{z}.
\end{aligned}$$

By variable change  $w = qz$  for the second term of the last line of (3.11), we have  $dw/w = dz/z$ , and the integral over  $\mathbb{T}$  changes to that over  $\mathbb{T}_q := \{w \in \mathbb{C} \mid |w| = |q|\}$ . Thus, from (3.11) we have

$$\langle \nabla_{q,z} \varphi(z) \rangle = \int_{\mathbb{T}} \Phi(z) F_-(z) \varphi(z) \frac{dz}{z} - \int_{\mathbb{T}_q} \Phi(z) F_-(z) \varphi(z) \frac{dz}{z}.$$

By the Cauchy theorem, in order to complete the proof, it suffices to show that the function  $\Phi(z) F_-(z) \varphi(z) z^{-1}$  is holomorphic on the annulus  $\{z \in \mathbb{C} \mid |q| \leq |z| \leq 1\}$ . From (3.5) and (3.6), we have

$$\begin{aligned}
(3.12) \quad \Phi(z) F_-(z) \varphi(z) z^{-1} &= \frac{(z^2, z^{-2}; q)_\infty}{\prod_{k=1}^4 (a_k z, a_k z^{-1}; q)_\infty} \frac{1}{z^{-1}} \frac{\prod_{k=1}^4 (1 - a_k z^{-1})}{1 - z^{-2}} \varphi(z) z^{-1} \\
&= \frac{(z^2, qz^{-2}; q)_\infty}{\prod_{k=1}^4 (a_k z, qa_k z^{-1}; q)_\infty} \varphi(z).
\end{aligned}$$

Since  $\varphi(z)$  is holomorphic on  $\mathbb{C}^*$  by assumption, poles of the function  $\Phi(z) F_-(z) \varphi(z) z^{-1}$  coincide with zero points of the denominator of (3.12). The set of solutions  $z$  of  $(a_k z, qa_k z^{-1}; q)_\infty = 0$  are expressed as

$$\{q^{-n} a_k^{-1} \mid n = 0, 1, 2, \dots\} \cup \{q^{1+n} a_k \mid n = 0, 1, 2, \dots\}.$$

Since  $|a_k| < 1$  and  $|q| < 1$  by assumption, we have

$$|q^{-n} a_k^{-1}| = \frac{1}{|q|^n |a_k|} > 1, \quad |q^{1+n} a_k| = |q|^{1+n} |a_k| < |q|.$$

This implies that no poles of the function  $\Phi(z) F_-(z) \varphi(z) z^{-1}$  are included in the annulus  $\{z \in \mathbb{C} \mid |q| \leq |z| \leq 1\}$ . This completes the proof.  $\square$

**3.2.  $q$ -Difference equations for the Askey–Wilson integral.** For the proof of Proposition 3.1 we first show the  $q$ -difference equation for  $I(a_1, a_2, a_3, a_4)$  as follows:

**Proposition 3.4.** *Suppose that  $|a_k| < 1$  ( $k = 1, 2, 3, 4$ ). The integral  $I(a_1, a_2, a_3, a_4)$  satisfies*

$$(3.13) \quad \frac{I(qa_1, a_2, a_3, a_4)}{I(a_1, a_2, a_3, a_4)} = \frac{(1 - a_1 a_2)(1 - a_1 a_3)(1 - a_1 a_4)}{1 - a_1 a_2 a_3 a_4}.$$



**Proof.** Setting  $\varphi(z) = 1$ ,  $\nabla_{q,z}\varphi(z)$  is calculated as

$$\begin{aligned}
\nabla_{q,z}\varphi(z) &= F_-(z) + F_+(z) = \frac{1}{z^{-1}} \frac{\prod_{k=1}^4 (1 - a_k z^{-1})}{1 - z^{-2}} + \frac{1}{z} \frac{\prod_{k=1}^4 (1 - a_k z)}{1 - z^2} \\
&= \frac{1}{z - z^{-1}} \left[ z^2 (1 - a_1 z^{-1})(1 - a_2 z^{-1})(1 - a_3 z^{-1})(1 - a_4 z^{-1}) \right. \\
&\quad \left. - z^{-2} (1 - a_1 z)(1 - a_2 z)(1 - a_3 z)(1 - a_4 z) \right] \\
(3.14) \quad &= \frac{1}{z - z^{-1}} \left[ z^2 (1 - z^{-1}E_1 + z^{-2}E_2 - z^{-3}E_3 + z^{-4}E_4) \right. \\
&\quad \left. - z^{-2} (1 - zE_1 + z^2E_2 - z^3E_3 + z^4E_4) \right] \\
&= \frac{(z^2 - z^{-2})(1 - E_4) - (z - z^{-1})(E_1 - E_3)}{z - z^{-1}} \\
&= (z + z^{-1})(1 - E_4) - (E_1 - E_3),
\end{aligned}$$

where  $E_r$  ( $r = 0, 1, \dots, 4$ ) is the  $r$ th elementary symmetric polynomial of  $a_1, \dots, a_4$ . From (3.14), using  $e(a_1, z)$  defined by (2.3),  $\nabla_{q,z}\varphi(z)$  is expanded as

$$(3.15) \quad \nabla_{q,z}\varphi(z) = C_1 e(a_1, z) + C_0$$

where the coefficients  $C_0$  and  $C_1$  are independent of  $z$ . We now determine  $C_0$  and  $C_1$ . Comparing the highest degree polynomials of (3.14) and (3.15), we have  $-C_1 = 1 - E_4$ , so that  $C_1 = -(1 - a_1 a_2 a_3 a_4)$ . On the other hand, since  $e(a_1, a_1) = 0$ , if we put  $z = a_1$  in (3.15), then  $C_0 = \nabla_{q,z}\varphi(a_1)$ . From  $F_-(a_1) = 0$  we also have  $\nabla_{q,z}\varphi(a_1) = F_+(a_1)$ . Therefore we obtain

$$\begin{aligned}
C_0 &= \nabla_{q,z}\varphi(a_1) = F_+(a_1) = \frac{1}{a_1} \frac{\prod_{k=1}^4 (1 - a_k a_1)}{1 - a_1^2} \\
&= a_1^{-1} (1 - a_1 a_2)(1 - a_1 a_3)(1 - a_1 a_4).
\end{aligned}$$

From (3.10) of  $q$ -Stokes' theorem we have

$$0 = \langle \nabla_{q,z}\varphi(z) \rangle = C_1 \langle e(a_1, z) \rangle + C_0 \langle 1 \rangle,$$

so that

$$\langle e(a_1, z) \rangle = -\frac{C_0}{C_1} \langle 1 \rangle = \frac{(1 - a_1 a_2)(1 - a_1 a_3)(1 - a_1 a_4)}{a_1 (1 - a_1 a_2 a_3 a_4)} \langle 1 \rangle.$$

Here we have  $I(a_1, a_2, a_3, a_4) = \langle 1 \rangle$  and  $I(qa_1, a_2, a_3, a_4) = \langle a_1 e(a_1, z) \rangle = a_1 \langle e(a_1, z) \rangle$  because

$$\frac{T_{q,a_1}\Phi(z)}{\Phi(z)} = \frac{(a_1 z, a_1 z^{-1}; q)_\infty}{(qa_1 z, qa_1 z^{-1}; q)_\infty} = (1 - a_1 z)(1 - a_1 z^{-1}) = a_1 e(a_1, z).$$

Therefore we obtain

$$I(qa_1, a_2, a_3, a_4) = \frac{(1 - a_1 a_2)(1 - a_1 a_3)(1 - a_1 a_4)}{1 - a_1 a_2 a_3 a_4} I(a_1, a_2, a_3, a_4),$$

which completes the proof.  $\square$

By repeated use of Proposition 3.4 we have the following.

**Proposition 3.5.**  $I(a_1, a_2, a_3, a_4)$  coincides with  $P(a_1, a_2, a_3, a_4)$  up to a multiplicative constant, i.e.,

$$(3.16) \quad I(a_1, a_2, a_3, a_4) = cP(a_1, a_2, a_3, a_4).$$

where  $c$  is some constant independent of  $a_1, a_2, a_3, a_4$ .

**Proof.** From (3.4)  $P(a_1, a_2, a_3, a_4)$  satisfies the same  $q$ -difference equation as (3.13), and (3.13) is symmetric with respect to  $a_1, a_2, a_3, a_4$ . This implies

$$(3.17) \quad \frac{I(a_1, a_2, a_3, a_4)}{P(a_1, a_2, a_3, a_4)} = \frac{I(qa_1, qa_2, qa_3, qa_4)}{P(qa_1, qa_2, qa_3, qa_4)} = \lim_{N \rightarrow \infty} \frac{I(q^N a_1, q^N a_2, q^N a_3, q^N a_4)}{P(q^N a_1, q^N a_2, q^N a_3, q^N a_4)}.$$

From (3.4) we have

$$(3.18) \quad \lim_{N \rightarrow \infty} P(q^N a_1, q^N a_2, q^N a_3, q^N a_4) = 1.$$

On the other hand, if we set  $c$  as

$$(3.19) \quad c = \lim_{N \rightarrow \infty} I(q^N a_1, q^N a_2, q^N a_3, q^N a_4) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z^2, z^{-2}; q)_{\infty} \frac{dz}{z},$$

then, using (3.17) and (3.18) we obtain (3.16).  $\square$

In order to compute the constant  $c$  in (3.16) as  $c = 2/(q; q)_{\infty}$  we want to know a special value of the integral  $I(a_1, a_2, a_3, a_4)$ . In the following section, we show several special values of  $I(a_1, a_2, a_3, a_4)$  at specific points.

**3.3. Special values of the Askey–Wilson integral.** In this subsection, we show special values of  $I(a_1, a_2, a_3, a_4)$  at four specific points

$$(a_1, a_2, a_3, a_4) = (0, 0, 0, 0), (0, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}), (1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}) \text{ and } (0, 0, q^{\frac{1}{2}}, -q^{\frac{1}{2}}).$$

The evaluation at  $(a_1, a_2, a_3, a_4) = (1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}})$  is the most simplest case.

**Lemma 3.6.**

$$I(1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{dz}{z} = 1, \quad P(1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}) = \frac{(q; q)_{\infty}}{2}.$$

**Proof.** Using (2.1) we have

$$I(1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{(z^2, z^{-2}; q)_{\infty}}{(z, z^{-1}, -z, -z^{-1}, q^{\frac{1}{2}}z, q^{\frac{1}{2}}z^{-1}, -q^{\frac{1}{2}}z, -q^{\frac{1}{2}}z^{-1}; q)_{\infty}} \frac{dz}{z} = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{dz}{z},$$

which is equal to 1 due to Cauchy's residue theorem. On the other hand, by definition (3.4) we have

$$\begin{aligned} P(1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}) &= \frac{(q; q)_{\infty}}{(-1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q^{\frac{1}{2}}, -q; q)_{\infty}} = \frac{2(q; q)_{\infty}}{(-1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}; q)_{\infty}^2} \\ &= \frac{(q; q)_{\infty}}{2(-q, q^{\frac{1}{2}}, -q^{\frac{1}{2}}; q)_{\infty}^2} = \frac{(q; q)_{\infty}}{2} \quad (\text{from (2.2)}), \end{aligned}$$

which completes the proof.  $\square$

Using (3.19) we can actually compute the constant  $c$  in Proposition 3.5 directly as

**Lemma 3.7.**

$$I(0, 0, 0, 0) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z^2, z^{-2}; q)_{\infty} \frac{dz}{z} = \frac{2}{(q; q)_{\infty}}, \quad P(0, 0, 0, 0) = 1.$$

**Proof.** Using Jacobi's theta function  $\vartheta(x; q)$  defined as (2.4), we have

$$\begin{aligned} \frac{(q; q)_{\infty}}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z^2, z^{-2}; q)_{\infty} \frac{dz}{z} &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (1 - z^{-2})z^{-1}(z^2, qz^{-2}, q; q)_{\infty} dz \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z^{-1} - z^{-3})\vartheta(z^2; q) dz = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z^{-1} - z^{-3}) \sum_{n=-\infty}^{\infty} (-z^2)^n q^{\binom{n}{2}} dz. \end{aligned}$$

For the integrand of the above expression, the point  $z = 0$  is the unique essential singularity, and its residue is calculated from the Laurent expansion

$$\begin{aligned} (z^{-1} - z^{-3}) \sum_{n=-\infty}^{\infty} (-z^2)^n q^{\binom{n}{2}} &= (z^{-1} - z^{-3})(\dots + q^3 z^{-4} - qz^{-2} + 1 - z^2 + qz^4 - \dots) \\ &= (\dots + q^3 z^{-5} - qz^{-3} + z^{-1} - z + qz^3 - \dots) \\ &\quad - (\dots + q^3 z^{-8} - qz^{-6} + z^{-3} - z^{-1} + qz - \dots), \end{aligned}$$

whose residue (coefficient of  $z^{-1}$ ) is 2. Therefore Cauchy's residue theorem implies

$$\frac{(q; q)_{\infty}}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z^2, z^{-2}; q)_{\infty} \frac{dz}{z} = 2,$$

which shows  $I(0, 0, 0, 0)$  in Lemma 3.7. From (3.18) we have  $P(0, 0, 0, 0) = 1$ .  $\square$

The evaluations at  $(a_1, a_2, a_3, a_4) = (0, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}})$  and  $(0, 0, q^{\frac{1}{2}}, -q^{\frac{1}{2}})$  are almost the same as that at  $(0, 0, 0, 0)$ .

**Lemma 3.8.**

$$(3.20) \quad \begin{aligned} I(0, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}) &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z, z^{-1}; q)_{\infty} \frac{dz}{z} = \frac{2}{(q; q)_{\infty}}, \\ P(0, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}) &= 1. \end{aligned}$$

**Proof.** From (2.1),  $I(0, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}})$  is expressed as

$$\begin{aligned} I(0, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}) &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{(z^2, z^{-2}; q)_{\infty}}{(-z, -z^{-1}, q^{\frac{1}{2}}z, q^{\frac{1}{2}}z^{-1}, -q^{\frac{1}{2}}z, -q^{\frac{1}{2}}z^{-1}; q)_{\infty}} \frac{dz}{z} \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{(z, -z, q^{\frac{1}{2}}z, -q^{\frac{1}{2}}z, z^{-1}, -z^{-1}, q^{\frac{1}{2}}z^{-1}, -q^{\frac{1}{2}}z^{-1}; q)_{\infty}}{(-z, -z^{-1}, q^{\frac{1}{2}}z, q^{\frac{1}{2}}z^{-1}, -q^{\frac{1}{2}}z, -q^{\frac{1}{2}}z^{-1}; q)_{\infty}} \frac{dz}{z} \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z, z^{-1}; q)_{\infty} \frac{dz}{z}. \end{aligned}$$

From (2.5) we have

$$\begin{aligned} \frac{(q; q)_{\infty}}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z, z^{-1}; q)_{\infty} \frac{dz}{z} &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (1 - z^{-1})z^{-1}(z, qz^{-1}, q; q)_{\infty} dz \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z^{-1} - z^{-2})\vartheta(z; q) dz = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z^{-1} - z^{-2}) \sum_{n=-\infty}^{\infty} (-z)^n q^{\binom{n}{2}} dz, \end{aligned}$$

whose integrand has the unique essential singularity at  $z = 0$ , and is expanded as

$$\begin{aligned} (z^{-1} - z^{-2}) \sum_{n=-\infty}^{\infty} (-z)^n q^{\binom{n}{2}} &= (z^{-1} - z^{-2})(\cdots + q^3 z^{-2} - qz^{-1} + 1 - z + qz^2 - \cdots) \\ &= (\cdots + q^3 z^{-3} - qz^{-2} + z^{-1} - 1 + qz - \cdots) \\ &\quad - (\cdots + q^3 z^{-4} - qz^{-3} + z^2 - z^{-1} + q - \cdots). \end{aligned}$$

The residue of the above expansion is 2 and the Cauchy's residue theorem implies

$$\frac{(q; q)_{\infty}}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z, z^{-1}; q)_{\infty} \frac{dz}{z} = 2,$$

which is equivalent to (3.20). Besides,  $P(0, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}) = (0; q)_{\infty} / (-q^{\frac{1}{2}}, q^{\frac{1}{2}}, -q; q)_{\infty} = 1$ .  $\square$

**Lemma 3.9.**

$$(3.21) \quad I(0, 0, q^{\frac{1}{2}}, -q^{\frac{1}{2}}) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z^2, z^{-2}; q^2)_{\infty} \frac{dz}{z} = \frac{2}{(q^2; q^2)_{\infty}},$$

$$(3.22) \quad P(0, 0, q^{\frac{1}{2}}, -q^{\frac{1}{2}}) = \frac{1}{(-q; q)_{\infty}},$$

and  $I(0, 0, q^{\frac{1}{2}}, -q^{\frac{1}{2}}) / P(0, 0, q^{\frac{1}{2}}, -q^{\frac{1}{2}}) = 2 / (q; q)_{\infty}$ .

**Proof.** From (2.1),  $I(0, 0, q^{\frac{1}{2}}, -q^{\frac{1}{2}})$  is expressed as

$$\begin{aligned} I(0, 0, q^{\frac{1}{2}}, -q^{\frac{1}{2}}) &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{(z^2, z^{-2}; q)_{\infty}}{(q^{\frac{1}{2}}z, q^{\frac{1}{2}}z^{-1}, -q^{\frac{1}{2}}z, -q^{\frac{1}{2}}z^{-1}; q)_{\infty}} \frac{dz}{z} \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{(z, -z, q^{\frac{1}{2}}z, -q^{\frac{1}{2}}z, z^{-1}, -z^{-1}, q^{\frac{1}{2}}z^{-1}, -q^{\frac{1}{2}}z^{-1}; q)_{\infty}}{(q^{\frac{1}{2}}z, q^{\frac{1}{2}}z^{-1}, -q^{\frac{1}{2}}z, -q^{\frac{1}{2}}z^{-1}; q)_{\infty}} \frac{dz}{z} \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z, -z, z^{-1}, -z^{-1}; q)_{\infty} \frac{dz}{z} = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z^2, z^{-2}; q^2)_{\infty} \frac{dz}{z}. \end{aligned}$$

Using (2.4) for  $\vartheta(x; q^2)$  we have

$$\begin{aligned} \frac{(q^2; q^2)_{\infty}}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z^2, z^{-2}; q^2)_{\infty} \frac{dz}{z} &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (1 - z^{-2})z^{-1}(z^2, q^2z^{-2}, q^2; q^2)_{\infty} dz \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z^{-1} - z^{-3})\vartheta(z^2; q^2) dz = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z^{-1} - z^{-3}) \sum_{n=-\infty}^{\infty} (-z^2)^n q^{\binom{n}{2}} dz \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z^{-1} - z^{-3}) \sum_{n=-\infty}^{\infty} (-z^2)^n q^{n(n-1)} dz, \end{aligned}$$

whose integrand has the unique essential singularity at  $z = 0$ , and is expanded as

$$\begin{aligned} (z^{-1} - z^{-3}) \sum_{n=-\infty}^{\infty} (-z^2)^n q^{n(n-1)} &= (z^{-1} - z^{-3})(\cdots + q^6 z^{-4} - q^2 z^{-2} + 1 - z^2 + q^2 z^4 - \cdots) \\ &= (\cdots + q^6 z^{-5} - q^2 z^{-3} + z^{-1} - z + q^2 z^3 - \cdots) \\ &\quad - (\cdots + q^6 z^{-8} - q^2 z^{-6} + z^{-3} - z^{-1} + q^2 z - \cdots). \end{aligned}$$

The residue of the above expansion is 2 and the Cauchy's residue theorem implies

$$\frac{(q^2; q^2)_\infty}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z^2, z^{-2}; q^2)_\infty \frac{dz}{z} = 2,$$

which is equivalent to (3.21). On the other hand, by definition we have (3.22). Thus,

$$\frac{I(0, 0, q^{\frac{1}{2}}, -q^{\frac{1}{2}})}{P(0, 0, q^{\frac{1}{2}}, -q^{\frac{1}{2}})} = \frac{2/(q^2; q^2)_\infty}{1/(-q; q)_\infty} = \frac{2(-q; q)_\infty (q; q)_\infty}{(q^2; q^2)_\infty (q; q)_\infty} = \frac{2(q^2; q^2)_\infty}{(q^2; q^2)_\infty (q; q)_\infty} = \frac{2}{(q; q)_\infty},$$

which completes the proof.  $\square$

We conclude this section with the following.

**Proof of Proposition 3.1.** Since the constant  $c$  of (3.16) in Proposition 3.5 is independent of  $a_1, a_2, a_3, a_4$ , for any case of Lemmas 3.6–3.9 the constant  $c$  equals

$$\frac{I(0, 0, 0, 0)}{P(0, 0, 0, 0)} = \frac{I(0, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}})}{P(0, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}})} = \frac{I(1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}})}{P(1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}})} = \frac{I(0, 0, q^{\frac{1}{2}}, -q^{\frac{1}{2}})}{P(0, 0, q^{\frac{1}{2}}, -q^{\frac{1}{2}})} = \frac{2}{(q; q)_\infty}.$$

This was the claim of Proposition 3.1.  $\square$

#### 4. NASSRALLAH–RAHMAN INTEGRAL

The aim of this section is to provide a way to prove the following identity of the Nassrallah–Rahman integral.

**Proposition 4.1.** *Suppose that  $a_k \in \mathbb{C}^*$  ( $1 \leq k \leq 5$ ) satisfy  $|a_k| < 1$ . Under the condition  $a_1 a_2 a_3 a_4 a_5 a_6 = q$ , we have*

$$(4.1) \quad \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{(z^2, z^{-2}, qa_6^{-1}z, qa_6^{-1}z^{-1}; q)_\infty}{\prod_{k=1}^5 (a_k z, a_k z^{-1}; q)_\infty} \frac{dz}{z} = \frac{2}{(q; q)_\infty} \frac{\prod_{i=1}^5 (qa_6^{-1}a_i^{-1}; q)_\infty}{\prod_{1 \leq i < j \leq 5} (a_i a_j; q)_\infty}.$$

Throughout this section, we define the function  $\Phi(z)$  on  $\mathbb{C}^*$  by

$$(4.2) \quad \Phi(z) := \frac{(z^2, z^{-2}, qa_6^{-1}z, qa_6^{-1}z^{-1}; q)_\infty}{\prod_{k=1}^5 (a_k z, a_k z^{-1}; q)_\infty},$$

and also denote by  $J(a_1, a_2, a_3, a_4, a_5, a_6)$  the left-hand side of (4.1). i.e.,

$$(4.3) \quad J(a_1, a_2, a_3, a_4, a_5, a_6) := \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \Phi(z) \frac{dz}{z}.$$

On the other hand, we define  $Q(a_1, a_2, a_3, a_4, a_5, a_6)$  by the infinite product

$$(4.4) \quad Q(a_1, a_2, a_3, a_4, a_5, a_6) := \frac{\prod_{i=1}^5 (qa_6^{-1}a_i^{-1}; q)_\infty}{\prod_{1 \leq i < j \leq 5} (a_i a_j; q)_\infty}.$$

##### 4.1. Definition of $q$ -difference operator $\nabla_{q,z}$ .

**Lemma 4.2.** *Let  $F_+(z)$  and  $F_-(z)$  be functions specified by*

$$(4.5) \quad F_+(z) := \frac{1}{z^2} \frac{\prod_{k=1}^6 (1 - a_k z)}{1 - z^2} = -\frac{1}{z - z^{-1}} z^{-3} \prod_{k=1}^6 (1 - a_k z),$$

$$(4.6) \quad F_-(z) := F_+(z^{-1}) = z^2 \frac{\prod_{k=1}^6 (1 - a_k z^{-1})}{1 - z^{-2}}.$$

Then, it follows that

$$\frac{T_{q,z}\Phi(z)}{\Phi(z)} = -\frac{F_+(z)}{T_{q,z}F_-(z)}.$$

**Proof.** By the definition (4.2) of  $\Phi(z)$  we have

$$\begin{aligned} \frac{\Phi(qz)}{\Phi(z)} &= \frac{(q^2z^2, q^{-2}z^{-2}; q)_\infty (q^2a_6^{-1}z, a_6^{-1}z^{-1}; q)_\infty}{(z^2, z^{-2}; q)_\infty (qa_6^{-1}z, qa_6^{-1}z^{-1}; q)_\infty} \frac{\prod_{k=1}^5 (a_kz, a_kz^{-1}; q)_\infty}{\prod_{k=1}^5 (qa_kz, q^{-1}a_kz^{-1}; q)_\infty} \\ &= \frac{(1 - q^{-2}z^{-2})(1 - q^{-1}z^{-2})}{(1 - z^2)(1 - qz^2)} \frac{1 - a_6^{-1}z^{-1}}{1 - qa_6^{-1}z} \prod_{k=1}^5 \frac{1 - a_kz}{1 - q^{-1}a_kz^{-1}} \\ &= -(q^{-1}z^{-2})^2 \frac{1 - q^{-2}z^{-2}}{1 - z^2} \prod_{k=1}^6 \frac{1 - a_kz}{1 - q^{-1}a_kz^{-1}} = -\frac{F_+(z)}{F_-(qz)}, \end{aligned}$$

which completes the proof.  $\square$

For an arbitrary meromorphic function  $\varphi(z)$  on  $\mathbb{C}^*$  we define the symbol

$$\langle \varphi(z) \rangle := \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \varphi(z)\Phi(z) \frac{dz}{z}$$

**Proposition 4.3.** *For an arbitrary meromorphic function  $\varphi(z)$  on  $\mathbb{C}^*$ , let  $\nabla_{q,z}$  be operator specified by*

$$(4.7) \quad (\nabla_{q,z}\varphi)(z) := F_-(z)\varphi(z) + F_+(z)T_{q,z}\varphi(z).$$

*Suppose that  $|a_k| < 1$  ( $k = 1, \dots, 5$ ). For an arbitrary holomorphic function  $\varphi(z)$  on  $\mathbb{C}^*$ , it follows that*

$$(4.8) \quad \langle \nabla_{q,z}\varphi(z) \rangle = 0.$$

**Proof.** The argument of the proof is paralleled with that of Proposition 3.3 and we omit the details.  $\square$

**4.2.  $q$ -Difference equations for the Nassrallah–Rahman integral.** In this subsection we explain a derivation of the  $q$ -difference equations for the Nassrallah–Rahman integral given as follows:

**Proposition 4.4.** *Suppose that  $|a_k| < 1$  ( $k = 1, \dots, 5$ ). Under the condition*

$$a_1a_2a_3a_4a_5a_6 = q,$$

*the integral  $J(a_1, \dots, a_6)$  satisfies*

$$(4.9) \quad J(qa_1, a_2, a_3, a_4, a_5, q^{-1}a_6) = J(a_1, a_2, a_3, a_4, a_5, a_6) \prod_{k=2}^5 \frac{1 - a_1a_k}{1 - qa_k^{-1}a_6^{-1}}.$$

This proposition is equivalent to the following.

**Proposition 4.5.** *Suppose that  $|a_k| < 1$  ( $k = 1, \dots, 5$ ). Under the condition*

$$a_1a_2a_3a_4a_5a_6 = 1,$$

*the integral  $J(a_1, \dots, a_6)$  satisfies*

$$(4.10) \quad J(qa_1, a_2, a_3, a_4, a_5, a_6) = J(a_1, a_2, a_3, a_4, a_5, qa_6) \prod_{k=2}^5 \frac{1 - a_1a_k}{1 - a_k^{-1}a_6^{-1}},$$

which is equivalent to

$$(4.11) \quad \langle e(a_1, z) \rangle = \langle e(a_6, z) \rangle \frac{a_6^2}{a_1^2} \prod_{k=2}^5 \frac{1 - a_1 a_k}{1 - a_k a_6}.$$

**Remark.** If we consider the substitution  $a_6 \rightarrow q^{-1}a_6$  for Proposition 4.5, then the balancing condition changes as  $a_1 a_2 a_3 a_4 a_5 a_6 = 1 \rightarrow a_1 a_2 a_3 a_4 a_5 (q^{-1}a_6) = 1$ , i.e.,  $a_1 a_2 a_3 a_4 a_5 a_6 = q$  and (4.10) becomes (4.9). We prove Proposition 4.5 instead of Proposition 4.4.

**Proof.** We will prove (4.11) in Lemma 4.6. Here we just confirm the equivalence between (4.10) and (4.11). Since  $T_{q, a_1} \Phi(z)/\Phi(z) = a_1 e(a_1, z)$  and  $T_{q, a_6} \Phi(z)/\Phi(z) = a_6^{-1} e(a_6, z)$  by definition, we have

$$J(qa_1, a_2, a_3, a_4, a_5, a_6) = a_1 \langle e(a_1, z) \rangle, \quad J(a_1, a_2, a_3, a_4, a_5, qa_6) = a_6^{-1} \langle e(a_6, z) \rangle,$$

so that

$$(4.12) \quad \frac{J(qa_1, a_2, a_3, a_4, a_5, a_6)}{J(a_1, a_2, a_3, a_4, a_5, qa_6)} = a_1 a_6 \frac{\langle e(a_1, z) \rangle}{\langle e(a_6, z) \rangle}.$$

On the other hand, under the condition  $a_1 a_2 a_3 a_4 a_5 a_6 = 1$ , we have

$$(4.13) \quad \prod_{k=2}^5 \frac{1 - a_1 a_k}{1 - a_k^{-1} a_6^{-1}} = \frac{1}{a_1 a_2 a_3 a_4 a_5 a_6} \prod_{k=2}^5 \frac{1 - a_1 a_k}{1 - a_k^{-1} a_6^{-1}} = \frac{a_6^3}{a_1} \prod_{k=2}^5 \frac{1 - a_1 a_k}{1 - a_k a_6}.$$

(4.12) and (4.13) implies the equivalence between (4.10) and (4.11) □

**Lemma 4.6.** *Suppose that  $|a_k| < 1$  ( $k = 1, \dots, 5$ ). Under the condition*

$$a_1 a_2 a_3 a_4 a_5 a_6 = 1,$$

we have

$$C_1 \langle e(a_1, z) \rangle + C_6 \langle e(a_6, z) \rangle = 0,$$

where the coefficients  $C_1$  and  $C_6$  are given as

$$(4.14) \quad C_1 = \frac{a_1}{a_6(a_6 - a_1)} \prod_{k=2}^5 (1 - a_k a_6),$$

$$(4.15) \quad C_6 = \frac{a_6}{a_1(a_1 - a_6)} \prod_{k=2}^5 (1 - a_1 a_k).$$

**Proof.** Taking  $\varphi(z)$  in (4.7) as  $\varphi(z) = 1$ ,  $\nabla_{q, z} \varphi(z)$  is written as

$$\begin{aligned} \nabla_{q, z} \varphi(z) &= F_-(z) + F_+(z) \\ &= \frac{1}{z^{-2}} \frac{\prod_{k=1}^6 (1 - a_k z^{-1})}{1 - z^{-2}} + \frac{1}{z^2} \frac{\prod_{k=1}^6 (1 - a_k z)}{1 - z^2} \\ &= \frac{1}{z - z^{-1}} \left( \frac{1}{z^{-3}} \prod_{k=1}^6 (1 - a_k z^{-1}) - \frac{1}{z^3} \prod_{k=1}^6 (1 - a_k z) \right) \\ &= \frac{1}{z - z^{-1}} \left( S_0 z^3 - S_1 z^2 + S_2 z - S_3 + S_4 z^{-1} - S_5 z^{-2} + S_6 z^{-3} \right. \\ &\quad \left. - S_0 z^{-3} + S_1 z^{-2} - S_2 z^{-1} + S_3 - S_4 z + S_5 z^2 - S_6 z^3 \right) \end{aligned}$$

$$(4.16) \quad = (S_0 - S_6)(z^2 + z^{-2}) + (S_5 - S_1)(z + z^{-1}) + (S_0 + S_2 - S_4 - S_6),$$

where  $S_r$  ( $r = 0, 1, \dots, 6$ ) is the  $r$ th elementary symmetric polynomial of  $a_1, \dots, a_6$ . The condition  $a_1 a_2 a_3 a_4 a_5 a_6 = 1$  implies  $S_0 - S_6 = 0$ . From (4.16), we have

$$\nabla_{q,z}\varphi(z) = (S_5 - S_1)(z + z^{-1}) + (S_0 + S_2 - S_4 - S_6) \in \mathbb{C}(z + z^{-1}) \oplus \mathbb{C}1.$$

Since the set  $\{e(a_1, z), e(a_6, z)\}$  forms a  $\mathbb{C}$ -basis of the space  $\mathbb{C}(z + z^{-1}) \oplus \mathbb{C}1$ , we can expand  $\nabla_{q,z}\varphi(z)$  as

$$(4.17) \quad \nabla_{q,z}\varphi(z) = C_1 e(a_1, z) + C_6 e(a_6, z),$$

where  $C_1$  and  $C_6$  are some constants independent of  $z$ . Thus we can write

$$(4.18) \quad F_-(z) + F_+(z) = C_1 e(a_1, z) + C_6 e(a_6, z).$$

Since  $e(a_1, a_1) = 0$  and  $F_-(a_1) = 0$ , (4.18) with  $z = a_1$  implies

$$F_+(a_1) = C_6 e(a_6, a_1).$$

In the same way, (4.18) with  $z = a_6$  implies  $F_+(a_6) = C_1 e(a_1, a_6)$ . Thus we obtain

$$C_1 = \frac{F_+(a_6)}{e(a_1, a_6)} = \frac{a_1}{a_6(a_6 - a_1)} \prod_{k=2}^5 (1 - a_k a_6),$$

$$C_6 = \frac{F_+(a_1)}{e(a_6, a_1)} = \frac{a_6}{a_1(a_1 - a_6)} \prod_{k=2}^5 (1 - a_1 a_k).$$

Applying Proposition 4.3 to (4.17), we obtain

$$C_1 \langle e(a_1, z) \rangle + C_6 \langle e(a_6, z) \rangle = \langle \nabla_{q,z}\varphi(z) \rangle = 0,$$

which completes the proof.  $\square$

**Proposition 4.7.** *Suppose that  $|a_k| < 1$  ( $k = 1, \dots, 5$ ). Under the condition*

$$a_1 a_2 a_3 a_4 a_5 a_6 = q,$$

*$J(a_1, \dots, a_6)$  coincides with  $Q(a_1, \dots, a_6)$  up to a multiplicative constant, i.e.,*

$$(4.19) \quad J(a_1, a_2, a_3, a_4, a_5, a_6) = c Q(a_1, a_2, a_3, a_4, a_5, a_6),$$

*where  $c$  is some constant independent of  $a_1, \dots, a_6$ .*

**Proof.** By the definition (4.4),  $Q(a_1, \dots, a_6)$  satisfies the same  $q$ -difference equation as (4.9), i.e.,

$$(4.20) \quad Q(qa_1, a_2, a_3, a_4, a_5, q^{-1}a_6) = Q(a_1, a_2, a_3, a_4, a_5, a_6) \prod_{k=2}^5 \frac{1 - a_1 a_k}{1 - qa_k^{-1} a_6^{-1}}.$$

Thus, under the condition  $a_1 a_2 a_3 a_4 a_5 a_6 = q$ , considering the ratio of (4.9) and (4.20),

$$\frac{J(a_1, a_2, a_3, a_4, a_5, a_6)}{Q(a_1, a_2, a_3, a_4, a_5, a_6)} = \frac{J(qa_1, a_2, a_3, a_4, a_5, q^{-1}a_6)}{Q(qa_1, a_2, a_3, a_4, a_5, q^{-1}a_6)},$$



which is symmetric with respect to  $a_1, \dots, a_5$ . Therefore we obtain

$$\begin{aligned}
(4.21) \quad \frac{J(a_1, a_2, a_3, a_4, a_5, a_6)}{Q(a_1, a_2, a_3, a_4, a_5, a_6)} &= \frac{J(qa_1, qa_2, qa_3, qa_4, qa_5, q^{-5}a_6)}{Q(qa_1, qa_2, qa_3, qa_4, qa_5, q^{-5}a_6)} \\
&= \frac{J(q^N a_1, q^N a_2, q^N a_3, q^N a_4, q^N a_5, q^{-5N} a_6)}{Q(q^N a_1, q^N a_2, q^N a_3, q^N a_4, q^N a_5, q^{-5N} a_6)} \\
&= \lim_{N \rightarrow \infty} \frac{J(q^N a_1, q^N a_2, q^N a_3, q^N a_4, q^N a_5, q^{-5N} a_6)}{Q(q^N a_1, q^N a_2, q^N a_3, q^N a_4, q^N a_5, q^{-5N} a_6)}.
\end{aligned}$$

From (4.4), we have

$$\begin{aligned}
(4.22) \quad &\lim_{N \rightarrow \infty} Q(q^N a_1, q^N a_2, q^N a_3, q^N a_4, q^N a_5, q^{-5N} a_6) \\
&= \lim_{N \rightarrow \infty} \frac{\prod_{i=1}^5 (q^{1+4N} a_6^{-1} a_i^{-1}; q)_\infty}{\prod_{1 \leq i < j \leq 5} (q^{2N} a_i a_j; q)_\infty} = 1,
\end{aligned}$$

and putting  $c$  as  $c = \lim_{N \rightarrow \infty} J(q^N a_1, q^N a_2, q^N a_3, q^N a_4, q^N a_5, q^{-5N} a_6)$  we obtain

$$\begin{aligned}
(4.23) \quad c &= \lim_{N \rightarrow \infty} \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{(z^2, z^{-2}, q^{1+5N} a_6^{-1} z, q^{1+5N} a_6^{-1} z^{-1}; q)_\infty}{\prod_{i=1}^5 (q^N a_i z, q^N a_i z^{-1}; q)_\infty} \frac{dz}{z} \\
&= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z^2, z^{-2}; q)_\infty \frac{dz}{z}.
\end{aligned}$$

Therefore, (4.21), (4.22) and (4.23) imply (4.19).  $\square$

**Proof of proposition 4.1.** Since we already knew that

$$\frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z^2, z^{-2}; q)_\infty \frac{dz}{z} = \frac{2}{(q; q)_\infty}$$

from Lemma 3.7, the constant  $c$  as (4.23) is equal to  $2/(q; q)_\infty$ .  $\square$

**4.3. Special values of the Nassrallah–Rahman integral.** We can find the special values that are simply computed.

**Lemma 4.8.**

$$\begin{aligned}
J(1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, a_5, a_5^{-1}) &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{dz}{(1 - a_5 z)(z - a_5)} = \frac{1}{1 - a_5^2}, \\
Q(1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, a_5, a_5^{-1}) &= \frac{(q; q)_\infty}{2(1 - a_5^2)}.
\end{aligned}$$

**Proof.** When  $a_1 = 1$ ,  $a_2 = -1$ ,  $a_3 = q^{\frac{1}{2}}$ ,  $a_4 = -q^{\frac{1}{2}}$ , the balancing condition  $a_1 a_2 a_3 a_4 a_5 a_6 = q$  implies  $a_5 a_6 = 1$  Using  $a_6 = a_5^{-1}$  the integral  $J(a_1, \dots, a_6)$  is computed as

$$\begin{aligned}
&J(1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, a_5, a_5^{-1}) \\
&= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{(z^2, z^{-2}, qa_5 z, qa_5 z^{-1}; q)_\infty}{(z, z^{-1}, -z, -z^{-1}, q^{\frac{1}{2}} z, q^{\frac{1}{2}} z^{-1}, -q^{\frac{1}{2}} z, -q^{\frac{1}{2}} z^{-1}, a_5 z, a_5 z^{-1}; q)_\infty} \frac{dz}{z} \\
&= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{1}{(1 - a_5 z)(1 - a_5 z^{-1})} \frac{dz}{z} = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{dz}{(1 - a_5 z)(z - a_5)}
\end{aligned}$$

$$\begin{aligned}
&= \operatorname{Res}_{z=a_5} \frac{dz}{(1-a_5z)(z-a_5)} \quad (\text{by Cauchy's residue theorem with } |a_5| < 1) \\
&= \lim_{z \rightarrow a_5} \frac{1}{1-a_5z} = \frac{1}{1-a_5^2}.
\end{aligned}$$

On the other hand, from (4.4),  $Q(1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, a_5, a_5^{-1})$  is computed as

$$\begin{aligned}
Q(1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, a_5, a_5^{-1}) &= \frac{(qa_5, -qa_5, q^{\frac{1}{2}}a_5, -q^{\frac{1}{2}}a_5, q; q)_\infty}{(-1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, a_5, -q^{\frac{1}{2}}, q^{\frac{1}{2}}, -a_5, -q, q^{\frac{1}{2}}a_5, -q^{\frac{1}{2}}a_5; q)_\infty} \\
&= \frac{(qa_5, -qa_5, q; q)_\infty}{(-1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, a_5, -q^{\frac{1}{2}}, q^{\frac{1}{2}}, -a_5, -q; q)_\infty} = \frac{(qa_5, -qa_5, q; q)_\infty}{2(a_5, -a_5; q)_\infty (-q, q^{\frac{1}{2}}, -q^{\frac{1}{2}}; q)_\infty^2} \\
&= \frac{(q; q)_\infty}{2(1-a_5)(1+a_5)}, \quad (\text{from (2.2)})
\end{aligned}$$

which completes the proof.  $\square$

The constant  $c$  in Proposition 4.7 is also obtained from

$$c = \frac{J(1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, a_5, a_5^{-1})}{Q(1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, a_5, a_5^{-1})} = \frac{1/(1-a_5^2)}{(q; q)_\infty/2(1-a_5^2)} = \frac{2}{(q; q)_\infty},$$

which also completes the proof of Proposition 4.1.

**4.4. The relation between the Askey–Wilson and Nassrallah–Rahman integrals.** Since the Askey–Wilson integral  $I(a_1, \dots, a_4)$  is obtained from the Nassrallah–Rahman integral  $J(a_1, \dots, a_6)$  as a limiting case of  $a_5 \rightarrow 0$ , we can understand  $I(a_1, a_2, a_3, a_4)$  is a special value of  $J(a_1, \dots, a_6)$ . Conversely  $J(a_1, \dots, a_6)$  is obtained from  $I(a_1, \dots, a_4)$  using the following relation. (See [7] for the  $BC_n$  case.)

**Proposition 4.9.** *Suppose that  $|a_k| < 1$  ( $k = 1, \dots, 5$ ) and the condition*

$$a_1 a_2 a_3 a_4 a_5 a_6 = q.$$

*Then,*

$$(4.24) \quad J(a_1, a_2, a_3, a_4, a_5, a_6) = I(a_1, a_2, a_3, a_4) \prod_{k=1}^4 \frac{(qa_k^{-1}a_6^{-1}; q)_\infty}{(a_k a_5; q)_\infty}.$$

**Proof.** Using the recurrence relation (4.9) for  $J(a_1, \dots, a_6)$  repeatedly we obtain

$$\begin{aligned}
J(a_1, a_2, a_3, a_4, a_5, a_6) &= J(a_1, a_2, a_3, a_4, qa_5, q^{-1}a_6) \prod_{k=1}^4 \frac{1 - qa_k^{-1}a_6^{-1}}{1 - a_k a_5} \\
&= J(a_1, a_2, a_3, a_4, q^N a_5, q^{-N} a_6) \prod_{k=1}^4 \frac{(qa_k^{-1}a_6^{-1}; q)_N}{(a_k a_5; q)_N} \\
&= \lim_{N \rightarrow \infty} J(a_1, a_2, a_3, a_4, q^N a_5, q^{-N} a_6) \prod_{k=1}^4 \frac{(qa_k^{-1}a_6^{-1}; q)_\infty}{(a_k a_5; q)_\infty},
\end{aligned}$$

where we also have

$$\lim_{N \rightarrow \infty} J(a_1, a_2, a_3, a_4, q^N a_5, q^{-N} a_6)$$

$$\begin{aligned}
&= \lim_{N \rightarrow \infty} \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{(z^2, z^{-2}, q^{1+N}a_6^{-1}z, q^{1+N}a_6^{-1}z^{-1}; q)_{\infty}}{(q^N a_5 z, q^N a_5 z^{-1}; q)_{\infty} \prod_{k=1}^4 (a_k z, a_k z^{-1}; q)_{\infty}} \frac{dz}{z} \\
&= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{(z^2, z^{-2}; q)_{\infty}}{\prod_{k=1}^4 (a_k z, a_k z^{-1}; q)_{\infty}} \frac{dz}{z} = I(a_1, a_2, a_3, a_4),
\end{aligned}$$

which implies (4.24).  $\square$

## 5. ROOT SYSTEM OF TYPE $G_2$

**5.1. Root system of type  $G_2$  and its Weyl group  $W$ .** In this subsection, we follow [10] for the basic terminology of the root system of type  $G_2$  and its Weyl group  $W$ .

Let  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  be the standard basis of  $\mathbb{R}^3$  with the inner product  $(\cdot, \cdot)$  satisfying  $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$ , and let  $V$  be the hyperplane in  $\mathbb{R}^3$  with equation  $\xi_1 + \xi_2 + \xi_3 = 0$ , i.e.,  $V = \{\xi \in \mathbb{R}^3 \mid (\xi, \varepsilon_1 + \varepsilon_2 + \varepsilon_3) = 0\}$ .

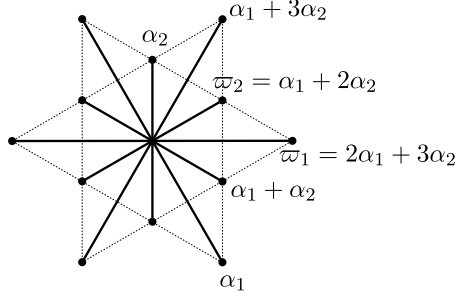


FIGURE 1. Root system  $R$

Let  $R \subset V$  be the root system of type  $G_2$  given by

$$R = \{\pm\bar{\varepsilon}_1, \pm\bar{\varepsilon}_2, \pm\bar{\varepsilon}_3\} \cup \{\pm(\bar{\varepsilon}_1 - \bar{\varepsilon}_2), \pm(\bar{\varepsilon}_1 - \bar{\varepsilon}_3), \pm(\bar{\varepsilon}_2 - \bar{\varepsilon}_3)\},$$

where  $\bar{\varepsilon}_i = \varepsilon_i - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)/3$ . We refer the setting of the root system of type  $G_2$  to Macdonald's book [12]. We fix the set of simple roots  $\{\alpha_1, \alpha_2\} \subset R$  given by

$$\alpha_1 = \bar{\varepsilon}_1 - \bar{\varepsilon}_2 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \bar{\varepsilon}_2 = (-\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3)/3.$$

The set of positive roots is given by

$$\begin{aligned}
R^+ &= \{\bar{\varepsilon}_1, \bar{\varepsilon}_2, -\bar{\varepsilon}_3\} \cup \{\bar{\varepsilon}_1 - \bar{\varepsilon}_2, \bar{\varepsilon}_1 - \bar{\varepsilon}_3, \bar{\varepsilon}_2 - \bar{\varepsilon}_3\} \\
&= \{\alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\} \cup \{\alpha_1, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}.
\end{aligned}$$

We also fix the set of fundamental weights  $\{\varpi_1, \varpi_2\}$  by  $(\alpha_i^\vee, \varpi_j) = \delta_{ij}$ , where  $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ . This implies that

$$\varpi_1 = 2\alpha_1 + 3\alpha_2, \quad \varpi_2 = \alpha_1 + 2\alpha_2.$$

Let  $P$  and  $Q$  be the weight lattice and root lattice defined by  $P = \mathbb{Z}\varpi_1 + \mathbb{Z}\varpi_2$  and  $Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ , respectively. For the root system  $G_2$ , the root lattice  $Q$  coincides with the weight lattice  $P$ .

Let  $W$  be the Weyl group of type  $G_2$  generated by orthogonal reflections  $s_\alpha$  ( $\alpha \in R$ ) with respect to the hyperplane perpendicular to  $\alpha \in R$ , which are given by  $s_\alpha(\xi) = \xi - (\alpha^\vee, \xi)\alpha$ . The group  $W$  is generated by the reflections  $s_i = s_{\alpha_i} : V \rightarrow V$  ( $i = 1, 2$ ), and is isomorphic to the dihedral group of order 12. Moreover,  $W$  is explicitly written as

$$(5.1) \quad W = \{(s_1 s_2)^k, (s_1 s_2)^k s_2 \mid k = 0, 1, \dots, 5\},$$

The element  $w_0 = (s_1 s_2)^3$  is the *longest element* of  $W$ . Note also that the inner product and the reflections are uniquely extended linearly to  $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ .

We fix the set of fundamental coweights  $\{\omega_1, \omega_2\}$  by  $(\omega_i, \alpha_j) = \delta_{ij}$ , so that  $\omega_1 = \varpi_1$ ,  $\omega_2 = 3\varpi_2$ . Let  $P^\vee$  be the coweight lattice defined by  $P^\vee = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ . For  $c \in \mathbb{C}$  and  $\omega \in P^\vee$  we denote by  $S_{c,\omega}$  the  $c$ -shift operator with respect to  $\omega$  for functions  $f(\zeta)$  on  $V_{\mathbb{C}}$  by

$$(5.2) \quad S_{c,\omega} f(\zeta) = f(\zeta + c\omega).$$

We also define action of the Weyl group  $W$  on  $f(\zeta)$  by

$$(5.3) \quad w.f(\zeta) = f(w^{-1}\zeta) \quad (w \in W).$$

We consider the mapping from  $V_{\mathbb{C}}$  to  $(\mathbb{C}^*)^2$  by

$$(5.4) \quad \zeta \mapsto z = (e^{2\pi\sqrt{-1}(\zeta, \alpha_1)}, e^{2\pi\sqrt{-1}(\zeta, \alpha_2)}).$$

If we write  $\zeta \in V_{\mathbb{C}}$  with the fundamental coweights by  $\zeta = \zeta_1\omega_1 + \zeta_2\omega_2$ , then the above mapping is written as  $\zeta \mapsto z = (e^{2\pi\sqrt{-1}\zeta_1}, e^{2\pi\sqrt{-1}\zeta_2})$ . For  $\lambda \in P$ , we write  $z^\lambda = e^{2\pi\sqrt{-1}(\zeta, \lambda)}$ . In particular, for  $\lambda = \lambda_1\alpha_1 + \lambda_2\alpha_2 \in Q = P$  we have the expression  $z^\lambda = z_1^{\lambda_1} z_2^{\lambda_2}$ , where  $z_i = z^{\alpha_i}$ . Through (5.3) and (5.4), for  $w \in W$  we can define  $w.z^\lambda = z^{w\lambda}$ , i.e.,

$$w.z^\lambda = w.e^{2\pi\sqrt{-1}(\zeta, \lambda)} = e^{2\pi\sqrt{-1}(w^{-1}\zeta, \lambda)} = e^{2\pi\sqrt{-1}(\zeta, w\lambda)} = z^{w\lambda},$$

and we can also define  $w.f(z)$  for functions  $f(z) = f(z_1, z_2)$  on  $(\mathbb{C}^*)^2$  as

$$w.f(z) = f(w.z_1, w.z_2) = f(z^{w\alpha_1}, z^{w\alpha_2}),$$

so that, for instance, we have

$$(5.5) \quad s_1.f(z_1, z_2) = f(z_1^{-1}, z_1 z_2), \quad s_2.f(z_1, z_2) = f(z_1 z_2^3, z_2^{-1}),$$

and

$$(5.6) \quad w_0.f(z_1, z_2) = (s_1 s_2)^3.f(z_1, z_2) = f(z_1^{-1}, z_2^{-1}).$$

We say that a function  $f(z)$  is  $W$ -symmetric if  $w.f(z) = f(z)$  for all  $w \in W$ , and that  $f(z)$  is  $W$ -skew symmetric if  $w.f(z) = (\text{sgn } w)f(z)$  for all  $w \in W$ . By chain rule for differential forms, we have

$$(5.7) \quad \begin{aligned} \frac{d(s_1.z_1)}{s_1.z_1} &= -\frac{dz_1}{z_1}, \quad \frac{d(s_1.z_2)}{s_1.z_2} = \frac{dz_1}{z_1} + \frac{dz_2}{z_2} \\ \frac{d(s_2.z_1)}{s_2.z_1} &= \frac{dz_1}{z_1} + 3\frac{dz_2}{z_2}, \quad \frac{d(s_2.z_2)}{s_2.z_2} = -\frac{dz_2}{z_2}. \end{aligned}$$

We fix  $q = e^{2\pi\sqrt{-1}\tau}$ , where  $\text{Im } \tau > 0$ . If we consider a function  $f(z) = f(z_1, z_2)$  on  $(\mathbb{C}^*)^2$  as the function on  $V_{\mathbb{C}}$  through (5.4), then the  $q$ -shift operators for  $f(z)$  with respect to  $z_i$  ( $i = 1, 2$ )

$$T_{q,z_1}f(z) = f(qz_1, z_2), \quad T_{q,z_2}f(z) = f(z_1, qz_2)$$

are induced by the  $\tau$ -shift operators  $S_{\tau,\omega_i}$  with respect to  $\omega_i \in P^{\vee}$  ( $i = 1, 2$ ), respectively.

Using the notation  $x_i = e^{2\pi\sqrt{-1}(\zeta,\bar{\varepsilon}_i)}$  ( $i = 1, 2, 3$ ), we have

$$x_1x_2x_3 = e^{2\pi\sqrt{-1}(\zeta,\bar{\varepsilon}_1+\bar{\varepsilon}_2+\bar{\varepsilon}_3)} = 1$$

and the variable change  $(z_1, z_2) \mapsto (x_1, x_2)$  of  $(\mathbb{C}^*)^2$ , where

$$(5.8) \quad x_1 = z_1z_2, \quad x_2 = z_2 \quad \text{and} \quad \frac{dx_1}{x_1} = \frac{dz_1}{z_1} + \frac{dz_2}{z_2}, \quad \frac{dx_2}{x_2} = \frac{dz_2}{z_2}.$$

Though using the coordinates  $(x_1, x_2, x_3)$  with  $x_1x_2x_3 = 1$  instead of  $(z_1, z_2)$  we sometimes have simple expressions for functions on  $(\mathbb{C}^*)^2$  in appearance, like the integrand shown in the left-hand side of (1.6) for instance, we use the coordinates  $(z_1, z_2)$  of  $(\mathbb{C}^*)^2$  associated with simple roots in the succeeding sections.

**5.2.  $W$  invariant Laurent polynomials.** For a function  $f(z)$  on  $(\mathbb{C}^*)^2$  we define

$$\mathcal{A}f(z) := \sum_{w \in W} (\text{sgn } w)w.f(z)$$

which we call the  $W$ -skew symmetrization of  $f(z)$ . By definition  $\mathcal{A}f(z)$  is  $W$ -skew symmetric. The dominance ordering  $<$  on  $P$  is defined by

$$\mu \leq \lambda \iff \lambda - \mu \in Q_+ = \mathbb{N}\alpha_1 + \mathbb{N}\alpha_2,$$

for  $\mu, \lambda \in P$ , where  $\mathbb{N} = \{0, 1, 2, \dots\}$ . We set

$$P_+ := \{\lambda \in P \mid (\alpha_1, \lambda) \geq 0, (\alpha_2, \lambda) \geq 0\} = \mathbb{N}\varpi_1 + \mathbb{N}\varpi_2,$$

whose elements are called the *dominant weights*.

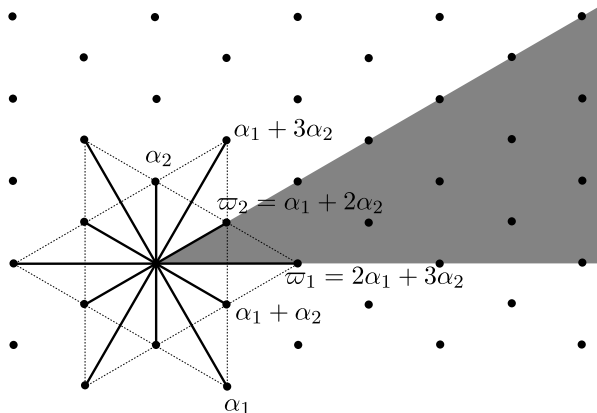


FIGURE 2. Dominant weights in  $P_+$

For  $\rho := \frac{1}{2} \sum_{\alpha \in R_+} \alpha = 3\alpha_1 + 5\alpha_2 = \varpi_1 + \varpi_2$ ,  $\mathcal{A}(z^\rho)$  satisfies

$$(5.9) \quad \mathcal{A}(z^\rho) = z_1^{-3}z_2^{-5}(1-z_2)(1-z_1z_2)(1-z_1z_2^2)(1-z_1)(1-z_1z_2^3)(1-z_1^2z_2^3),$$

which is called the *Weyl denominator*. For  $\lambda \in P_+$  we set

$$s_\lambda(z) := \frac{\mathcal{A}(z^{\rho+\lambda})}{\mathcal{A}(z^\rho)}, \quad m_\lambda(z) := \sum_{\mu \in W\lambda} z^\mu,$$

where  $W\lambda = \{w\lambda \mid w \in W\}$ . The functions  $s_\lambda(z)$  and  $m_\lambda(z)$  are  $W$ -invariant Laurent polynomials, and  $s_\lambda(z)$  are expanded as

$$s_\lambda(z) = m_\lambda(z) + \sum_{\substack{\mu \in P_+ \\ \mu < \lambda}} c_\mu m_\mu(z).$$

For instance, we see

$$\begin{aligned} s_0(z) &= 1, \\ s_{\varpi_2}(z) &= m_{\varpi_2}(z) + 1, \\ s_{\varpi_1}(z) &= m_{\varpi_1}(z) + m_{\varpi_2}(z) + 2, \\ s_{2\varpi_2}(z) &= m_{2\varpi_2}(z) + m_{\varpi_1}(z) + 2m_{\varpi_2}(z) + 3, \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} m_0(z) &= 1, \\ m_{\varpi_2}(z) &= z_2 + z_2^{-1} + z_1 z_2 + z_1^{-1} z_2^{-1} + z_1 z_2^2 + z_1^{-1} z_2^{-2}, \\ m_{\varpi_1}(z) &= z_1 + z_1^{-1} + z_1 z_2^3 + z_1^{-1} z_2^{-3} + z_1^2 z_2^3 + z_1^{-2} z_2^{-3}, \\ m_{2\varpi_2}(z) &= z_2^2 + z_2^{-2} + z_1^2 z_2^2 + z_1^{-2} z_2^{-2} + z_1^2 z_2^4 + z_1^{-2} z_2^{-4}, \\ &\vdots \end{aligned}$$

We denote by  $\mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]^W$  the set of  $W$ -invariant Laurent polynomials, which satisfies

$$\mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]^W = \bigoplus_{\lambda \in P_+} \mathbb{C} s_\lambda(z) = \bigoplus_{\lambda \in P_+} \mathbb{C} m_\lambda(z).$$

**Lemma 5.1.** *For  $n \in \mathbb{N}$ , let  $\mathcal{F}_n$  be  $\mathbb{C}$  vector space defined by*

$$(5.10) \quad \mathcal{F}_n := \bigoplus_{\substack{\lambda \in P_+ \\ \lambda \leq n\varpi_2}} \mathbb{C} m_\lambda(z).$$

*Then the dimension of  $\mathcal{F}_n$  as a  $\mathbb{C}$ -vector space is given as*

$$\dim_{\mathbb{C}} \mathcal{F}_{2m} = (m+1)^2, \quad \dim_{\mathbb{C}} \mathcal{F}_{2m+1} = (m+1)(m+2) \quad (m = 0, 1, 2, \dots).$$

For any  $a \in \mathbb{C}^*$  and any  $z = (z_1, z_2) \in (\mathbb{C}^*)^2$  we define  $g(a; z)$  by

$$(5.11) \quad \begin{aligned} g(a; z) &:= e(a, z_2) e(a, z_1 z_2) e(a, z_1 z_2^2) \\ &= a^{-3} (1 - a z_2) (1 - a z_2^{-1}) (1 - a z_1 z_2) \\ &\quad \times (1 - a z_1^{-1} z_2^{-1}) (1 - a z_1 z_2^2) (1 - a z_1^{-1} z_2^{-2}), \end{aligned}$$

where the symbol  $e(u, v)$  is by (2.3). The expansion of  $g(a; z)$  by  $m_\lambda(z)$  ( $\lambda \in P_+$ ) is given as

$$(5.12) \quad g(a; z) = -m_{2\varpi_2}(z) + (a + a^{-1})m_{\varpi_1}(z) - \left( (a + a^{-1})^2 - (a + a^{-1}) \right) m_{\varpi_2}(z) + \left( (a + a^{-1})^3 - 2 \right) m_0(z),$$

and the expansion of  $g(a; z)$  by  $s_\lambda(z)$  ( $\lambda \in P_+$ ) is given as

$$(5.13) \quad \begin{aligned} g(a; z) &= -s_{2\varpi_2}(z) + (a + a^{-1} + 1)s_{\varpi_1}(z) - \left( (a + a^{-1})^2 - 1 \right) s_{\varpi_2}(z) \\ &\quad + \left( (a + a^{-1})^3 + (a + a^{-1})^2 - 2(a + a^{-1}) - 2 \right) s_0(z) \\ &= -s_{2\varpi_2}(z) + \frac{(1 - a^3)}{a(1 - a)} s_{\varpi_1}(z) - \frac{(1 - a^6)}{a^2(1 - a^2)} s_{\varpi_2}(z) \\ &\quad + \frac{(1 - a^3)(1 - a^8)}{a^3(1 - a)(1 - a^4)} s_0(z). \end{aligned}$$

From (5.12), we see

$$g(a; z) \in \mathcal{F}_2, \quad g(a; z)g(b; z) \in \mathcal{F}_4$$

**5.3.  $\mathbb{C}$ -vector space  $\mathcal{F}_2$ .** By the definition (5.10) of  $\mathcal{F}_n$ , we have

$$\mathcal{F}_2 = \mathbb{C}m_0(z) \oplus \mathbb{C}m_{\varpi_2}(z) \oplus \mathbb{C}m_{\varpi_1}(z) \oplus \mathbb{C}m_{2\varpi_2}(z),$$

so that  $\dim_{\mathbb{C}} \mathcal{F}_2 = 4$ . Here we introduce two bases of the  $\mathbb{C}$ -vector space  $\mathcal{F}_2$ . For this purpose, we define two types of specific points as follows: Let  $p_{ij}, p_{ij}^*$  ( $1 \leq i < j \leq 6$ ) be points in  $(\mathbb{C}^*)^2$  specified by

$$(5.14) \quad p_{ij} := (a_i/a_j, a_j) \in (\mathbb{C}^*)^2,$$

(so that we have  $z_2 = a_j$ ,  $z_1 z_2 = a_i$ ,  $z_1 z_2^2 = a_i a_j$  if  $z = p_{ij}$ ) and

$$(5.15) \quad p_{ij}^* := (a_i^2/a_j, a_j/a_i) \in (\mathbb{C}^*)^2,$$

(so that we have  $z_2 = a_j/a_i$ ,  $z_1 z_2 = a_i$ ,  $z_1 z_2^2 = a_j$  if  $z = p_{ij}^*$ ).

**5.3.1. The basis  $\{e_0(z), e_{\varpi_2}(z), e_{\varpi_1}(z), e_{2\varpi_2}(z)\}$ .** We construct a basis of  $\mathbb{C}$ -vector space  $\mathcal{F}_2$ . We first define

$$\begin{aligned} e_0(z) &:= 1, \\ e_{\varpi_2}(z) &:= m_{\varpi_2}(z) - m_{\varpi_2}(p_{12}), \\ e_{2\varpi_2}(z) &:= -g(a_1; z) = m_{2\varpi_2}(z) - (a_1 + a_1^{-1})m_{\varpi_1}(z) + \cdots. \end{aligned}$$

Moreover we define  $e_{\varpi_1}(z)$  as

$$(5.16) \quad e_{\varpi_1}(z) := m_{\varpi_1}(z) + c_{\varpi_2} m_{\varpi_2}(z) + c_0,$$

where the coefficients  $c_{\varpi_2}$  and  $c_0$  are uniquely determined to be satisfied as

$$e_{\varpi_1}(p_{12}) = 0 \quad \text{and} \quad e_{\varpi_1}(p_{12}^*) = 0.$$

We remark that the coefficients  $c_{\varpi_2}$  and  $c_0$  are actually given as

$$c_{\varpi_2} := -\frac{m_{\varpi_1}(p_{12}) - m_{\varpi_1}(p_{12}^*)}{m_{\varpi_2}(p_{12}) - m_{\varpi_2}(p_{12}^*)}, \quad c_0 := \frac{m_{\varpi_1}(p_{12})m_{\varpi_2}(p_{12}^*) - m_{\varpi_1}(p_{12}^*)m_{\varpi_2}(p_{12})}{m_{\varpi_2}(p_{12}) - m_{\varpi_2}(p_{12}^*)}.$$

for  $e_{\varpi_1}(z)$ .

**Remark.** The values of  $e_{\varpi_1}(p_{1j})$  and  $e_{\varpi_1}(p_{1j}^*)$  are actually given

$$e_{\varpi_1}(p_{1j}) = e(a_2, a_j)e(a_2, a_1 a_j), \quad e_{\varpi_1}(p_{1j}^*) = e(a_2, a_j)e(a_2, a_1/a_j),$$

respectively. (We do not use these explicit forms in this paper.)

By definition the set  $\{e_0(z), e_{\varpi_2}(z), e_{\varpi_1}(z), e_{2\varpi_2}(z)\}$  is linearly independent.

	P12	P12*	P13	P23
$e_0(z)$	1	1	1	1
$e_{\varpi_2}(z)$	0	*	*	*
$e_{\varpi_1}(z)$	0	0	*	*
$e_{2\varpi_2}(z)$	0	0	0	*

In particular, since we have

$$(5.17) \quad e_{\varpi_2}(z) = m_{\varpi_2}(z) - m_{\varpi_2}(p_{12}) = s_{\varpi_2}(z) - s_{\varpi_2}(p_{12})$$

we can actually compute  $e_{\varpi_2}(p_{1j})$  and  $e_{\varpi_2}(p_{1j}^*)$  as

$$(5.18) \quad \begin{aligned} e_{\varpi_2}(p_{1j}) &= -(1 + a_1)(1 - a_2/a_j)(1 - a_1 a_2 a_j), \\ e_{\varpi_2}(p_{1j}^*) &= -a_1^{-1} a_2^{-1} (1 + a_1)(1 - a_1 a_2/a_j)(1 - a_2 a_j), \end{aligned}$$

which will be used in §6.3.

**Proposition 5.2.** *The set  $\{e_0(z), e_{\varpi_2}(z), e_{\varpi_1}(z), e_{2\varpi_2}(z)\}$  forms a basis of  $\mathbb{C}$ -vector space  $\mathcal{F}_2$ , i.e.,*

$$\mathcal{F}_2 = \mathbb{C}e_0(z) \oplus \mathbb{C}e_{\varpi_2}(z) \oplus \mathbb{C}e_{\varpi_1}(z) \oplus \mathbb{C}e_{2\varpi_2}(z).$$

In particular,  $\dim_{\mathbb{C}} \mathcal{F}_2 = 4$ .

### 5.3.2. The basis $\{g(a_1; z), g(a_2; z), g(a_3; z), G(z)\}$ of $\mathcal{F}_2$ .

**Proposition 5.3.** *suppose that  $a_1, \dots, a_4 \in \mathbb{C}^*$  satisfy  $a_i \neq a_j$  and  $a_i a_j \neq 1$  for  $1 \leq i < j \leq 4$ . Then, the set  $\{g(a_1; z), \dots, g(a_4; z)\}$  forms a basis of  $\mathbb{C}$ -vector space  $\mathcal{F}_2$ , i.e.,*

$$\mathcal{F}_2 = \mathbb{C}g(a_1; z) \oplus \mathbb{C}g(a_2; z) \oplus \mathbb{C}g(a_3; z) \oplus \mathbb{C}g(a_4; z).$$

In particular,  $\dim_{\mathbb{C}} \mathcal{F}_2 = 4$ .

**Proof.** Since we already know that  $g(a; z) \in \mathcal{F}_2$ , it suffices to prove that the set  $\{g(a_1; z), \dots, g(a_4; z)\}$  is linearly independent. From the expansion (5.12) of  $g(a; z)$  in terms of  $m_\lambda(z)$ , we have

$$\begin{bmatrix} g(a_1; z) \\ g(a_2; z) \\ g(a_3; z) \\ g(a_4; z) \end{bmatrix} = \begin{bmatrix} (a_1 + a_1^{-1})^3 - 2 & -(a_1 + a_1^{-1})^2 + (a_1 + a_1^{-1}) & a_1 + a_1^{-1} & -1 \\ (a_2 + a_2^{-1})^3 - 2 & -(a_2 + a_2^{-1})^2 + (a_2 + a_2^{-1}) & a_2 + a_2^{-1} & -1 \\ (a_3 + a_3^{-1})^3 - 2 & -(a_3 + a_3^{-1})^2 + (a_3 + a_3^{-1}) & a_3 + a_3^{-1} & -1 \\ (a_4 + a_4^{-1})^3 - 2 & -(a_4 + a_4^{-1})^2 + (a_4 + a_4^{-1}) & a_4 + a_4^{-1} & -1 \end{bmatrix} \begin{bmatrix} m_0(z) \\ m_{\varpi_2}(z) \\ m_{\varpi_1}(z) \\ m_{2\varpi_2}(z) \end{bmatrix}.$$

The determinant of the above transition matrix is computed as

$$\begin{vmatrix} (a_1 + a_1^{-1})^3 - 2 & -(a_1 + a_1^{-1})^2 + (a_1 + a_1^{-1}) & a_1 + a_1^{-1} & -1 \\ (a_2 + a_2^{-1})^3 - 2 & -(a_2 + a_2^{-1})^2 + (a_2 + a_2^{-1}) & a_2 + a_2^{-1} & -1 \\ (a_3 + a_3^{-1})^3 - 2 & -(a_3 + a_3^{-1})^2 + (a_3 + a_3^{-1}) & a_3 + a_3^{-1} & -1 \\ (a_4 + a_4^{-1})^3 - 2 & -(a_4 + a_4^{-1})^2 + (a_4 + a_4^{-1}) & a_4 + a_4^{-1} & -1 \end{vmatrix}$$



$$\begin{aligned}
&= \begin{vmatrix} (a_1 + a_1^{-1})^3 & (a_1 + a_1^{-1})^2 & a_1 + a_1^{-1} & 1 \\ (a_2 + a_2^{-1})^3 & (a_2 + a_2^{-1})^2 & a_2 + a_2^{-1} & 1 \\ (a_3 + a_3^{-1})^3 & (a_3 + a_3^{-1})^2 & a_3 + a_3^{-1} & 1 \\ (a_4 + a_4^{-1})^3 & (a_4 + a_4^{-1})^2 & a_4 + a_4^{-1} & 1 \end{vmatrix} \quad (\text{by elementary row operations}) \\
&= \prod_{1 \leq i < j \leq 4} (a_j + a_j^{-1} - a_i + a_i^{-1}) \quad (\text{by Vandermonde determinant}) \\
&= \prod_{1 \leq i < j \leq 4} e(a_i, a_j),
\end{aligned}$$

which is divisible by  $e(a_i, a_j)$  ( $1 \leq i < j \leq 4$ ). This implies that the above determinant does not vanish if and only if  $a_i \neq a_j$  and  $a_i a_j \neq 1$  for  $1 \leq i < j \leq 4$ . Therefore we obtain the fact that the set  $\{g(a_1; z), \dots, g(a_4; z)\}$  is linearly independent.  $\square$

Let  $p_{ij} \in (\mathbb{C}^*)^2$  be the point defined in (5.14). From the definition (5.11),  $g(a_k; z)$  is evaluated as

$$\begin{aligned}
g(a_k; p_{ij}) &= e(a_k, a_i) e(a_k, a_j) e(a_k, a_i a_j) \\
&= a_k^{-3} (1 - a_k a_i) (1 - a_k a_i^{-1}) (1 - a_k a_j) (1 - a_k a_j^{-1}) (1 - a_k a_i a_j) (1 - a_k a_i^{-1} a_j^{-1}),
\end{aligned}$$

so that we have

$$(5.19) \quad g(a_i; p_{ij}) = g(a_j; p_{ij}) = 0.$$

We now define the function  $G(z) \in \mathcal{F}_2$  as

$$(5.20) \quad G(z) := \frac{1}{\prod_{i=1}^3 e(a_4; a_i)} \left[ g(a_4; z) - \frac{g(a_4; p_{23})}{g(a_1; p_{23})} g(a_1; z) - \frac{g(a_4; p_{13})}{g(a_2; p_{13})} g(a_2; z) - \frac{g(a_4; p_{12})}{g(a_3; p_{12})} g(a_3; z) \right],$$

which will be used later.

**Remark.** The function  $G(z)$  seems to be very important for the integral of type  $G_2$ . In fact the elliptic version of the function  $G(z)$  plays an essential role in research of the elliptic Gustafson integral of type  $G_2$ . See [10].

**Lemma 5.4.** *The function  $G(z)$  satisfies*

$$(5.21) \quad G(p_{12}) = G(p_{13}) = G(p_{23}) = 0.$$

and

$$(5.22) \quad G(p_{14}) = \frac{(1 - a_1^2)(1 - a_2 a_4^{-1})(1 - a_3 a_4^{-1})(1 - a_1 a_2 a_4)(1 - a_1 a_3 a_4)}{(1 - a_1)(1 - a_2 a_3)(1 - a_1 a_2 a_3^{-1})(1 - a_1 a_3 a_2^{-1})(1 - a_1 a_2 a_3)}.$$

**Proof.** By the definition (5.20) of  $G(z)$  we immediately have (5.21). (5.22) is due to a direct calculation.  $\square$

**Remark.** For an arbitrary  $x \in \mathbb{C}^*$ , we can confirm

$$(5.23) \quad G(a_1/x, x) = \frac{(1 - a_1^2)(1 - a_2 x^{-1})(1 - a_3 x^{-1})(1 - a_1 a_2 x)(1 - a_1 a_3 x)}{(1 - a_1)(1 - a_2 a_3)(1 - a_1 a_2 a_3^{-1})(1 - a_1 a_3 a_2^{-1})(1 - a_1 a_2 a_3)}.$$

We see that (5.22) is only a special case corresponding to  $x = a_4$  of (5.23). This is something interesting since the definition of  $G(z)$  depends on  $a_4$ .

**Corollary 5.5.** Let  $p_{ij}^* \in (\mathbb{C}^*)^2$  be the point defined by (5.15). Then we have

$$(5.24) \quad g(a_1; p_{12}^*) = g(a_2; p_{12}^*) = 0$$

and

$$(5.25) \quad G(p_{12}^*) = G(p_{21}^*) = \frac{(1-a_1^2)(1-a_2^2)}{(1-a_1a_2a_3^{-1})(1-a_1a_2a_3)} = \frac{(1-a_1^2)(1-a_2^2)}{a_1a_2 e(a_1a_2, a_3)}.$$

**Proof.** By the definition (5.11) we can immediately confirm (5.24). (5.25) is a special case of (5.23) corresponding to  $x = a_2/a_1$ .  $\square$

From the arguments above the set  $\{g(a_1; z), g(a_2; z), g(a_3; z), G(z)\}$  also forms a basis of  $\mathbb{C}$ -vector space  $\mathcal{F}_2$ , i.e.,

$$\mathcal{F}_2 = \mathbb{C}g(a_1; z) \oplus \mathbb{C}g(a_2; z) \oplus \mathbb{C}g(a_3; z) \oplus \mathbb{C}G(z).$$

	P23	P13	P12	P12*
$g(a_1; z)$	*	0	0	0
$g(a_2; z)$	0	*	0	0
$g(a_3; z)$	0	0	*	*
$G(z)$	0	0	0	*

5.4.  **$\mathbb{C}$ -vector space  $\mathcal{F}_4$ .** By the definition of (5.10) of  $\mathcal{F}_n$ , we have

$$\begin{aligned} \mathcal{F}_4 = & \mathbb{C}m_0(z) \oplus \mathbb{C}m_{\varpi_2}(z) \oplus \mathbb{C}m_{\varpi_1}(z) \oplus \mathbb{C}m_{2\varpi_2}(z) \oplus \mathbb{C}m_{\varpi_1+\varpi_2}(z) \oplus \mathbb{C}m_{3\varpi_2}(z) \\ & \oplus \mathbb{C}m_{2\varpi_1}(z) \oplus \mathbb{C}m_{\varpi_1+2\varpi_2}(z) \oplus \mathbb{C}m_{4\varpi_2}(z), \end{aligned}$$

so that  $\dim_{\mathbb{C}} \mathcal{F}_4 = 9$ . Here we construct a basis of the  $\mathbb{C}$ -vector space  $\mathcal{F}_4$ .

We use the abbreviation

$$(5.26) \quad g_{ij}(z) = g(a_i; z)g(a_j; z).$$

We denote by  $G_l^{ijk}(z)$  the function  $G(z) \in \mathcal{F}_2$  substituting  $a_1 \rightarrow a_i, a_2 \rightarrow a_j, a_3 \rightarrow a_k, a_4 \rightarrow a_l$  in its definition (5.20), i.e.,

$$(5.27) \quad G_l^{ijk}(z) := \frac{1}{\prod_{m \in \{i,j,k\}} e(a_l; a_m)} \left[ g(a_l; z) - \frac{g(a_l; p_{jk})}{g(a_i; p_{jk})} g(a_i; z) - \frac{g(a_l; p_{ik})}{g(a_j; p_{ik})} g(a_j; z) - \frac{g(a_l; p_{ij})}{g(a_k; p_{ij})} g(a_k; z) \right].$$

As we saw in (5.21), we have

$$G_l^{ijk}(p_{ij}) = G_l^{ijk}(p_{ik}) = G_l^{ijk}(p_{jk}) = 0.$$

Moreover, if we set

$$(5.28) \quad \begin{aligned} G_1(z) &:= g(a_1; z)G_5^{234}(z), \\ G_2(z) &:= g(a_2; z)G_5^{134}(z), \\ G_3(z) &:= g(a_3; z)G_5^{124}(z) \end{aligned}$$

then  $G_1(z), G_2(z), G_3(z) \in \mathcal{F}_4$ , and by definition we have

$$G_1(p_{ij}) = G_2(p_{ij}) = G_3(p_{ij}) = 0 \quad (1 \leq i < j \leq 4).$$

From (5.24) and (5.25), we have

$$G_1(p_{23}^*) = g(a_1; p_{23}^*) G_5^{234}(p_{23}^*) = e(a_1, a_3/a_2) e(a_1, a_2) e(a_1, a_3) \frac{(1-a_2^2)(1-a_3^2)}{a_2 a_3 e(a_2 a_3, a_4)},$$

$$G_1(p_{13}^*) = 0, \quad G_1(p_{12}^*) = 0.$$

$$(5.29) \quad \begin{array}{c|cccccccccc} & p_{34} & p_{24} & p_{14} & p_{23} & p_{13} & p_{12} & p_{23}^* & p_{13}^* & p_{12}^* \\ \hline g_{12} & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ g_{13} & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ g_{23} & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ g_{14} & 0 & 0 & 0 & * & 0 & 0 & * & 0 & 0 \\ g_{24} & 0 & 0 & 0 & 0 & * & 0 & 0 & * & 0 \\ g_{34} & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & * \\ G_1 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\ G_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ G_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{array}$$

From the argument above we have the following:

**Lemma 5.6.** *The set  $\{g_{ij}(z) \mid 1 \leq i < j \leq 4\} \cup \{G_1(z), G_2(z), G_3(z)\}$  forms a basis of  $\mathbb{C}$ -vector space  $\mathcal{F}_4$ , i.e.,*

$$\mathcal{F}_4 = \bigoplus_{1 \leq i < j \leq 4} \mathbb{C} g_{ij}(z) \oplus \bigoplus_{i=1}^3 \mathbb{C} G_i(z).$$

In particular,  $\dim_{\mathbb{C}} \mathcal{F}_4 = 9$ .

## 6. GUSTAFSON INTEGRAL (ASKEY–WILSON INTEGRAL OF TYPE $G_2$ )

In this section we give an alternative proof of Proposition 1.1 for Gustafson's  $q$ -integral of type  $G_2$ .

**6.1. Gustafson integral of type  $G_2$ .** Let  $\Phi(z)$  be function of  $z = (z_1, z_2) \in (\mathbb{C}^*)^2$  defined by

$$(6.1) \quad \Phi(z) := \Phi_+(z) \Phi_+(z^{-1}),$$

where  $z^{-1} = (z_1^{-1}, z_2^{-1})$  and

$$\Phi_+(z) := \frac{(z_2, z_1 z_2, z_1 z_2^2, z_1, z_1 z_2^3, z_1^2 z_2^3; q)_{\infty}}{\prod_{k=1}^4 (a_k z_2, a_k z_1 z_2, a_k z_1 z_2^2; q)_{\infty}}.$$

We define  $I(a_1, a_2, a_3, a_4)$  as

$$I(a_1, a_2, a_3, a_4) := \frac{1}{(2\pi\sqrt{-1})^2} \iint_{\mathbb{T}^2} \Phi(z) \frac{dz_1}{z_1} \frac{dz_2}{z_2},$$

and also define  $P(a_1, a_2, a_3, a_4)$  as

$$(6.2) \quad \begin{aligned} & P(a_1, a_2, a_3, a_4) \\ & := \frac{(a_1^2 a_2^2 a_3^2 a_4^2; q)_{\infty}}{(a_1 a_2 a_3 a_4; q)_{\infty}} \prod_{i=1}^4 \frac{(a_i; q)_{\infty}}{(a_i^2; q)_{\infty}} \prod_{1 \leq i < j \leq 4} \frac{1}{(a_i a_j; q)_{\infty}} \prod_{1 \leq i < j < k \leq 4} \frac{1}{(a_i a_j a_k; q)_{\infty}}. \end{aligned}$$

Using  $I(a_1, a_2, a_3, a_4)$  and  $P(a_1, a_2, a_3, a_4)$  Proposition 1.1 is stated as the following

**Theorem 6.1** (Gustafson [5, 6]). *Suppose that  $a_k \in \mathbb{C}^*$  ( $1 \leq k \leq 4$ ) satisfy  $|a_k| < 1$ . Then,*

$$I(a_1, a_2, a_3, a_4) = \frac{12}{(q; q)_\infty^2} P(a_1, a_2, a_3, a_4).$$

The aim of this section is to give an alternative proof for the above theorem. One of key steps for our proof is the following.

**Proposition 6.2.**  $I(a_1, a_2, a_3, a_4)$  satisfy the following  $q$ -difference equation.

$$(6.3) \quad \frac{I(qa_1, a_2, a_3, a_4)}{I(a_1, a_2, a_3, a_4)} = \frac{(1+a_1)(1-qa_1^2) \prod_{i=2}^4 (1-a_1 a_i) \prod_{2 \leq j < k \leq 4} (1-a_1 a_j a_k)}{(1+a_1 a_2 a_3 a_4)(1-qa_1^2 a_2^2 a_3^2 a_4^2)}.$$

**Proof.** We state the detail later in §6.3 □

**Remark.** Proposition 6.2 is also obtained as a corollary of Theorem 7.2 for the Nassrallah–Rahman integral of type  $G_2$  defined in §7.1. See Remark of Theorem 7.2.

Once we had the recurrence relation (6.3) of Proposition 6.2, by repeated use of (6.3) we have the following.

**Proposition 6.3.**  $I(a_1, a_2, a_3, a_4)$  coincides with  $P(a_1, a_2, a_3, a_4)$  up to a multiplicative constant, i.e.,

$$(6.4) \quad I(a_1, a_2, a_3, a_4) = c P(a_1, a_2, a_3, a_4).$$

where  $c$  is some constant independent of  $a_1, a_2, a_3, a_4$ .

**Proof.** From (6.2),  $P(a_1, a_2, a_3, a_4)$  satisfies the same  $q$ -difference equation as (6.3), and (6.3) is symmetric with respect to  $a_1, a_2, a_3, a_4$ . This implies

$$(6.5) \quad \frac{I(a_1, a_2, a_3, a_4)}{P(a_1, a_2, a_3, a_4)} = \frac{I(qa_1, qa_2, qa_3, qa_4)}{P(qa_1, qa_2, qa_3, qa_4)} = \lim_{N \rightarrow \infty} \frac{I(q^N a_1, q^N a_2, q^N a_3, q^N a_4)}{P(q^N a_1, q^N a_2, q^N a_3, q^N a_4)}$$

From (6.2) we have

$$(6.6) \quad \lim_{N \rightarrow \infty} P(q^N a_1, q^N a_2, q^N a_3, q^N a_4) = 1$$

On the other hand, if we set  $c$  as

$$\begin{aligned} c &= \lim_{N \rightarrow \infty} I(q^N a_1, q^N a_2, q^N a_3, q^N a_4) \\ &= \frac{1}{(2\pi\sqrt{-1})^2} \iint_{\mathbb{T}^2} (z_2, z_1 z_2, z_1 z_2^2, z_1, z_1 z_2^3, z_1^2 z_2^3; q)_\infty \\ &\quad \times (z_2^{-1}, z_1^{-1} z_2^{-1}, z_1^{-1} z_2^{-2}, z_1^{-1}, z_1^{-1} z_2^{-3}, z_1^{-2} z_2^{-3}; q)_\infty \frac{dz_1}{z_1} \frac{dz_2}{z_2}, \end{aligned}$$

then, from (6.5) and (6.6), we obtain (6.4). □

It is necessary to compute a special value of  $I(a_1, a_2, a_3, a_4)$  for proving that  $c$  in (6.4) is equal to  $12/(q; q)_\infty^2$ . Here we compute  $I(1, 0, 0, 0)$ .

**Lemma 6.4.**

$$\begin{aligned} I(1, 0, 0, 0) &= \frac{1}{(2\pi\sqrt{-1})^2} \iint_{\mathbb{T}^2} (z_1, z_1 z_2^3, z_1^2 z_2^3; q)_\infty (z_1^{-1}, z_1^{-1} z_2^{-3}, z_1^{-2} z_2^{-3}; q)_\infty \frac{dz_1}{z_1} \frac{dz_2}{z_2} \\ &= \frac{6}{(q; q)_\infty^2}. \end{aligned}$$

**Proof.** We state the detail in §6.4. For its proof we use the triple product identity (2.5) of Jacobi's theta function.  $\square$

**Proof of Theorem 6.1.** From the definition (6.2) of  $P(a_1, a_2, a_3, a_4)$  we have

$$P(1, 0, 0, 0) = \frac{1}{(-1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}; q)_\infty} = \frac{1}{2(-q, q^{\frac{1}{2}}, -q^{\frac{1}{2}}; q)_\infty} = \frac{(q; q)_\infty}{2(q, -q, q^{\frac{1}{2}}, -q^{\frac{1}{2}}; q)_\infty} = \frac{1}{2}.$$

Combining this with Lemma 6.4, we can compute  $c$  in Proposition 6.3 as

$$c = \frac{I(1, 0, 0, 0)}{P(1, 0, 0, 0)} = \frac{6/(q; q)_\infty^2}{1/2} = \frac{12}{(q; q)_\infty^2}.$$

From (6.4) this concludes our proof of Theorem 6.1.  $\square$

## 6.2. $q$ -Stokes' theorem.

**Lemma 6.5.** *Let  $F_+(z)$  and  $F_-(z)$  be functions specified as*

$$F_+(z) := \frac{(z_1^2 z_2^3)^{-\frac{1}{2}} \prod_{k=1}^4 (1 - a_k z_1 z_2)(1 - a_k z_1 z_2^2)}{(1 - z_1 z_2)(1 - z_1 z_2^2)(1 - z_1)(1 - z_1 z_2^3)(1 - z_1^2 z_2^3)}$$

and

$$\begin{aligned} (6.7) \quad F_-(z) &:= F_+(z^{-1}) \\ &= \frac{(z_1^2 z_2^3)^{\frac{1}{2}} \prod_{k=1}^4 (1 - a_k z_1^{-1} z_2^{-1})(1 - a_k z_1^{-1} z_2^{-2})}{(1 - z_1^{-1} z_2^{-1})(1 - z_1^{-1} z_2^{-2})(1 - z_1^{-1})(1 - z_1^{-1} z_2^{-3})(1 - z_1^{-2} z_2^{-3})}. \end{aligned}$$

Then, it follows that

$$(6.8) \quad \frac{T_{q, z_1} \Phi(z)}{\Phi(z)} = -\frac{F_+(z)}{T_{q, z_1} F_-(z)}.$$

**Proof.** By the definition (6.1) of  $\Phi(z)$  we have

$$\begin{aligned}
\frac{T_{q,z_1}\Phi(z)}{\Phi(z)} &= \frac{(1 - (qz_1z_2^3)^{-1})(1 - (qz_1^2z_2^3)^{-1})(1 - (q^2z_1^2z_2^3)^{-1})}{(1 - z_1z_2^3)(1 - z_1^2z_2^3)(1 - qz_1^2z_2^3)} \\
&\quad \times \frac{(1 - (qz_1z_2)^{-1})(1 - (qz_1z_2^2)^{-1})(1 - (qz_1)^{-1})}{(1 - z_1z_2)(1 - z_1z_2^2)(1 - z_1)} \\
&\quad \times \prod_{k=1}^4 \frac{(1 - a_kz_1z_2)(1 - a_kz_1z_2^2)}{(1 - a_k(qz_1z_2)^{-1})(1 - a_k(qz_1z_2^2)^{-1})} \\
&= -(qz_1^2z_2^3)^{-1} \frac{(1 - (qz_1z_2)^{-1})(1 - (qz_1z_2^2)^{-1})}{(1 - z_1z_2)(1 - z_1z_2^2)} \\
&\quad \times \frac{(1 - (qz_1)^{-1})(1 - (qz_1z_2^3)^{-1})(1 - (q^2z_1^2z_2^3)^{-1})}{(1 - z_1)(1 - z_1z_2^3)(1 - z_1^2z_2^3)} \\
&\quad \times \prod_{k=1}^4 \frac{(1 - a_kz_1z_2)(1 - a_kz_1z_2^2)}{(1 - a_k(qz_1z_2)^{-1})(1 - a_k(qz_1z_2^2)^{-1})}.
\end{aligned}$$

This implies (6.8).  $\square$

We denote by  $\mathcal{M}((\mathbb{C}^*)^2)$  the  $\mathbb{C}$ -vector space of meromorphic functions on  $(\mathbb{C}^*)^2$ , and by  $\mathcal{O}((\mathbb{C}^*)^2)$  the  $\mathbb{C}$ -vector space of holomorphic functions on  $(\mathbb{C}^*)^2$ . For each function  $\varphi(z) \in z_2^{\frac{1}{2}}\mathcal{M}((\mathbb{C}^*)^2) = \{z_2^{\frac{1}{2}}f(z) \mid f(z) \in \mathcal{M}((\mathbb{C}^*)^2)\}$  we define the function  $\nabla\varphi(z)$  by

$$(\nabla\varphi)(z) := F_+(z)T_{q,z_1}^{\frac{1}{2}}\varphi(z) + F_-(z)T_{q,z_1}^{-\frac{1}{2}}\varphi(z) \in \mathcal{M}((\mathbb{C}^*)^2),$$

and  $\nabla_{\text{sym}}\varphi(z)$  by the symmetrization of  $\nabla\varphi(z)$ :

$$\begin{aligned}
(\nabla_{\text{sym}}\varphi)(z) &:= \sum_{w \in W} w.(\nabla\varphi(z)) \\
&= \sum_{w \in W} w. \left( F_+(z)T_{q,z_1}^{\frac{1}{2}}\varphi(z) + F_-(z)T_{q,z_1}^{-\frac{1}{2}}\varphi(z) \right) \in \mathcal{M}((\mathbb{C}^*)^2)^W,
\end{aligned}$$

where  $\mathcal{M}((\mathbb{C}^*)^2)^W$  denotes the  $\mathbb{C}$ -vector space of  $W$ -invariant meromorphic functions on  $(\mathbb{C}^*)^2$ . For  $f(z) \in \mathcal{M}((\mathbb{C}^*)^2)$  we use the notation

$$\langle f(z) \rangle = \frac{1}{(2\pi\sqrt{-1})^2} \iint_{\mathbb{T}^2} f(z)\Phi(z) \frac{dz_1}{z_1} \frac{dz_2}{z_2}.$$

In particular, we have  $\langle 1 \rangle = I(a_1, a_2, a_3, a_4)$  and  $\langle a_1^3g(a_1; z) \rangle = I(qa_1, a_2, a_3, a_4)$ .

**Lemma 6.6** ( $q$ -Stokes' theorem). *Suppose that  $|a_k| < 1$  ( $k = 1, 2, 3, 4$ ). Then, for  $\varphi(z) \in z_2^{\frac{1}{2}}\mathcal{O}((\mathbb{C}^*)^2)$  we have*

$$\langle \nabla\varphi(z) \rangle = 0,$$

and

$$\langle \nabla_{\text{sym}}\varphi(z) \rangle = 0.$$

**Proof.** From (6.8) we have

$$(6.9) \quad T_{q,z_1} \left( \Phi(z)F_-(z)T_{q,z_1}^{-\frac{1}{2}}\varphi(z) \right) = -\Phi(z)F_+(z)T_{q,z_1}^{\frac{1}{2}}\varphi(z),$$

and from (6.7) we have

$$(6.10) \quad \begin{aligned} \Phi(z)F_-(z)T_{q,z_1}^{-\frac{1}{2}}\varphi(z) &= z_1z_2^{\frac{3}{2}}T_{q,z_1}^{-\frac{1}{2}}\varphi(z)\Phi_+(z) \\ &\times \frac{(z_2^{-1}, qz_1^{-1}z_2^{-1}, qz_1^{-1}z_2^{-2}, qz_1^{-1}, qz_1^{-1}z_2^{-3}, qz_1^{-2}z_2^{-3}; q)_\infty}{\prod_{k=1}^4 (a_k z_2^{-1}, a_k q z_1^{-1} z_2^{-1}, a_k q z_1^{-1} z_2^{-2}; q)_\infty}. \end{aligned}$$

Since we have  $z_1z_2^{\frac{3}{2}}T_{q,z_1}^{-\frac{1}{2}}\varphi(z) \in \mathcal{O}((\mathbb{C}^*)^2)$  from  $\varphi(z) \in z_2^{\frac{1}{2}}\mathcal{O}((\mathbb{C}^*)^2)$ , if  $|a_k| < 1$  ( $k = 1, 2, 3, 4$ ) and  $z_2$  is fixed as  $|z_2| = 1$ , then the right-hand side of (6.10) as a function of  $z_1$  has no poles in the annulus  $|q| \leq |z_1| \leq 1$ . Thus, by Cauchy's integral theorem we have

$$(6.11) \quad \begin{aligned} &\frac{1}{(2\pi\sqrt{-1})^2} \iint_{\mathbb{T}^2} T_{q,z_1} \left( \Phi(z)F_-(z)T_{q,z_1}^{-\frac{1}{2}}\varphi(z) \right) \frac{dz_1}{z_1} \frac{dz_2}{z_2} \\ &= \frac{1}{(2\pi\sqrt{-1})^2} \int_{|z_2|=1} \left( \int_{|z_1|=|q|} \Phi(z)F_-(z)T_{q,z_1}^{-\frac{1}{2}}\varphi(z) \frac{dz_1}{z_1} \right) \frac{dz_2}{z_2} \\ &\quad \text{(from variable change } z'_1 = qz_1) \\ &= \frac{1}{(2\pi\sqrt{-1})^2} \int_{|z_2|=1} \left( \int_{|z_1|=1} \Phi(z)F_-(z)T_{q,z_1}^{-\frac{1}{2}}\varphi(z) \frac{dz_1}{z_1} \right) \frac{dz_2}{z_2} \\ &\quad \text{(from Cauchy's integral theorem)} \\ &= \frac{1}{(2\pi\sqrt{-1})^2} \iint_{\mathbb{T}^2} \Phi(z)F_-(z)T_{q,z_1}^{-\frac{1}{2}}\varphi(z) \frac{dz_1}{z_1} \frac{dz_2}{z_2}. \end{aligned}$$

Combining (6.9) with (6.11), we obtain

$$\langle \nabla\varphi(z) \rangle = \frac{1}{(2\pi\sqrt{-1})^2} \iint_{\mathbb{T}^2} \Phi(z) \left( F_+(z)T_{q,z_1}^{\frac{1}{2}}\varphi(z) + F_-(z)T_{q,z_1}^{-\frac{1}{2}}\varphi(z) \right) \frac{dz_1}{z_1} \frac{dz_2}{z_2} = 0.$$

Moreover, since  $\Phi(z)$  and  $\frac{dz_1}{z_1} \frac{dz_2}{z_2}$  are  $W$ -symmetric, we compute

$$\begin{aligned} \langle \nabla_{\text{sym}}\varphi(z) \rangle &= \frac{1}{(2\pi\sqrt{-1})^2} \iint_{\mathbb{T}^2} \Phi(z) \nabla_{\text{sym}}\varphi(z) \frac{dz_1}{z_1} \frac{dz_2}{z_2} \\ &= \frac{1}{(2\pi\sqrt{-1})^2} \iint_{\mathbb{T}^2} \Phi(z) \left( \sum_{w \in W} w \cdot \left( F_+(z)T_{q,z_1}^{\frac{1}{2}}\varphi(z) + F_-(z)T_{q,z_1}^{-\frac{1}{2}}\varphi(z) \right) \right) \frac{dz_1}{z_1} \frac{dz_2}{z_2} \\ &= \frac{1}{(2\pi\sqrt{-1})^2} \sum_{w \in W} \iint_{\mathbb{T}^2} w \cdot \left( \Phi(z) \left( F_+(z)T_{q,z_1}^{\frac{1}{2}}\varphi(z) + F_-(z)T_{q,z_1}^{-\frac{1}{2}}\varphi(z) \right) \right) \frac{dz_1}{z_1} \frac{dz_2}{z_2} \\ &= \frac{1}{(2\pi\sqrt{-1})^2} \sum_{w \in W} \iint_{w^{-1}\mathbb{T}^2} \Phi(z) \left( F_+(z)T_{q,z_1}^{\frac{1}{2}}\varphi(z) + F_-(z)T_{q,z_1}^{-\frac{1}{2}}\varphi(z) \right) \frac{dz_1}{z_1} \frac{dz_2}{z_2} \\ &= \frac{1}{(2\pi\sqrt{-1})^2} \sum_{w \in W} \iint_{\mathbb{T}^2} \Phi(z) \left( F_+(z)T_{q,z_1}^{\frac{1}{2}}\varphi(z) + F_-(z)T_{q,z_1}^{-\frac{1}{2}}\varphi(z) \right) \frac{dz_1}{z_1} \frac{dz_2}{z_2} \\ &= 0, \end{aligned}$$

which completes the proof.  $\square$

We set  $W_0 = \langle s_2, w_0 \rangle \subset W$ , where  $w_0 = (s_1s_2)^3$  is the longest element of  $W$ .

**Lemma 6.7.** *Suppose that  $\varphi(z)$  is a  $W_0$ -invariant function, i.e.,  $s_2.\varphi(z) = \varphi(z)$  and  $w_0.\varphi(z) = \varphi(z)$ . Then,  $\nabla_{\text{sym}}\varphi(z)$  is expressed as*

$$(6.12) \quad \nabla_{\text{sym}}\varphi(z) = 4 \sum_{k=0}^5 F_k(z)\varphi_k(z),$$

where  $F_k(z)$  and  $\varphi_k(z)$  ( $k = 0, 1, \dots, 5$ ) are given as

$$(6.13) \quad F_k(z) = (s_1 s_2)^k .F_+(z), \quad \varphi_k(z) = (s_1 s_2)^k .T_{q, z_1}^{\frac{1}{2}}\varphi(z).$$

**Proof.** From the definition (6.7) of  $F_-(z)$ , we have

$$F_-(z) = F_+(z^{-1}) = (s_1 s_2)^3 .F_+(z) = F_3(z).$$

Since  $\varphi(z)$  is an invariant with respect to  $w_0 = (s_1 s_2)^3 \in W$ , i.e.,  $\varphi(z^{-1}) = \varphi(z)$ , we have

$$T_{q, z_1}^{-\frac{1}{2}}\varphi(z) = \varphi(q^{-\frac{1}{2}}z_1, z_2) = \varphi(q^{\frac{1}{2}}z_1^{-1}, z_2^{-1}) = \varphi_0(z^{-1}) = (s_1 s_2)^3 \varphi_0(z) = \varphi_3(z),$$

so that  $\nabla\varphi(z)$  is expressed as

$$\nabla\varphi(z) = F_+(z)T_{q, z_1}^{\frac{1}{2}}\varphi(z) + F_-(z)T_{q, z_1}^{-\frac{1}{2}}\varphi(z) = F_0(z)\varphi_0(z) + F_3(z)\varphi_3(z).$$

Thus we obtain

$$\begin{aligned} \nabla_{\text{sym}}\varphi(z) &= \sum_{w \in W} w.(\nabla\varphi(z)) = \sum_{w \in W} w.(F_0(z)\varphi_0(z) + F_3(z)\varphi_3(z)) \\ &= 2 \sum_{w \in W} w.(F_0(z)\varphi_0(z)). \end{aligned}$$

Then, using the expression (5.1) of  $W$ ,

$$(6.14) \quad \nabla_{\text{sym}}\varphi(z) = 2 \sum_{k=0}^5 (s_1 s_2)^k .(F_0(z)\varphi_0(z) + s_2.(F_0(z)\varphi_0(z))).$$

Since  $\varphi(z)$  is an invariant with respect to  $s_2$ , i.e.,  $\varphi(z_1, z_2) = \varphi(z_1 z_2^3, z_2^{-1})$ , we have

$$s_2.\varphi_0(z) = s_2.\varphi(q^{\frac{1}{2}}z_1, z_2) = \varphi(q^{\frac{1}{2}}z_1 z_2^3, z_2^{-1}) = \varphi(q^{\frac{1}{2}}z_1, z_2) = \varphi_0(z).$$

Since  $F_0(z) = F_+(z)$  is also an invariant with respect to  $s_2$ , using this we see that (6.14) implies  $\nabla_{\text{sym}}\varphi(z) = 4 \sum_{k=0}^5 (s_1 s_2)^k .F_0(z)\varphi_0(z)$ .  $\square$

### 6.3. $q$ -Difference equations – Proof of Proposition 6.2.

**Lemma 6.8.** *For  $i = 0, 1$  it follows that*

$$(6.15) \quad \frac{\nabla_{\text{sym}}(z_2^{-\frac{1}{2}} + z_2^{\frac{1}{2}})(z_2 + z_2^{-1})^i}{4} \in \mathbb{C}m_0(z) \oplus \mathbb{C}m_{\varpi_2}(z).$$

The explicit expressions of (6.15) are given by

$$(6.16) \quad \frac{\nabla_{\text{sym}}(z_2^{-\frac{1}{2}} + z_2^{\frac{1}{2}})}{4} = (E_4 - E_0) \left[ (E_0 + E_4)s_{\varpi_2}(z) - (E_1 + E_2 + E_3) \right],$$

and

$$(6.17) \quad \frac{\nabla_{\text{sym}}(z_2^{-\frac{1}{2}} + z_2^{\frac{1}{2}})(z_2 + z_2^{-1})}{4} = (E_3 - E_1) \left[ (E_0 + E_4)s_{\varpi_2}(z) - (E_1 + E_2 + E_3) \right],$$



where  $E_r$  ( $r = 0, 1, \dots, 4$ ) is the  $r$ th elementary symmetric polynomial of  $a_1, \dots, a_4$  given by

$$(6.18) \quad E_r = \sum_{1 \leq i_1 < \dots < i_r \leq 4} a_{i_1} a_{i_2} \cdots a_{i_r},$$

i.e.,  $E_0 = 1$ ,  $E_1 = a_1 + a_2 + a_3 + a_4$ ,  $\dots$ ,  $E_4 = a_1 a_2 a_3 a_4$ .

**Proof.** The proof follows by direct computation and we omit the details.  $\square$

**Lemma 6.9.** Let  $e_{\varpi_2}(z)$  be the function given as (5.17). If we set

$$(6.19) \quad \varphi(z) = z_2^{-\frac{1}{2}}(1 + z_2)e(a_1, z_2),$$

then, we have

$$(6.20) \quad \frac{\nabla_{\text{sym}}\varphi(z)}{4} = B_2 e_{\varpi_2}(z) + B_1,$$

where the coefficients  $B_1$  and  $B_2$  are given as

$$(6.21) \quad B_2 = a_1^{-1}(1 - a_1 a_2)(1 - a_1 a_3)(1 - a_1 a_4)(1 + a_1 a_2 a_3 a_4),$$

and

$$(6.22) \quad B_1 = a_1^{-2} a_2^{-1} (1 - a_1 a_2)(1 - a_1 a_3)(1 - a_1 a_4) \\ \times (1 + a_1)(1 + a_2)(1 - a_1 a_2 a_3)(1 - a_1 a_2 a_4)$$

**Proof.** Since  $\varphi(z)$  is written as

$$\varphi(z) = z_2^{-\frac{1}{2}}(1 + z_2)e(a_1, z_2) = (z_2^{-\frac{1}{2}} + z_2^{\frac{1}{2}})(a_1 + a_1^{-1}) - (z_2^{-\frac{1}{2}} + z_2^{\frac{1}{2}})(z_2 + z_2^{-1}),$$

by (6.15) in Lemma 6.8, we have (6.20). Combining  $e_{\varpi_2}(z) = s_{\varpi_2}(z) - s_{\varpi_2}(\mathfrak{p}_{12})$  with (6.20), we have

$$\frac{\nabla_{\text{sym}}\varphi(z)}{4} = B_2 (s_{\varpi_2}(z) - s_{\varpi_2}(\mathfrak{p}_{12})) + B_1 = B_2 s_{\varpi_2}(z) + \dots$$

Comparing both sides of the above equation using (6.16) and (6.17), we obtain

$$B_2 = (E_0 + E_4)(E_0 - E_4)(a_1 + a_1^{-1}) - (E_0 + E_4)(E_3 - E_1) \\ = (E_0 + E_4) \left[ (E_0 - E_4)(a_1 + a_1^{-1}) - (E_3 - E_1) \right].$$

From the factorization

$$(E_0 - E_4)(a_1 + a_1^{-1}) - (E_3 - E_1) = a_1^{-1}(1 - a_1 a_2)(1 - a_1 a_3)(1 - a_1 a_4),$$

we therefore obtain (6.21).

On the other hand, since  $e_{\varpi_2}(\mathfrak{p}_{12}) = 0$ , we have  $B_1 = \nabla_{\text{sym}}\varphi(\mathfrak{p}_{12})/4$ . Using Lemma 6.7 with  $F_2(\mathfrak{p}_{12}) = F_3(\mathfrak{p}_{12}) = F_4(\mathfrak{p}_{12}) = F_5(\mathfrak{p}_{12}) = 0$  and  $\varphi_1(\mathfrak{p}_{12}) = 0$ , we obtain

$$B_1 = \frac{\nabla_{\text{sym}}\varphi(\mathfrak{p}_{12})}{4} = F_0(\mathfrak{p}_{12})\varphi_0(\mathfrak{p}_{12}) = F_+(\mathfrak{p}_{12})\varphi(\mathfrak{p}_{12}) \\ = a_1^{-1} a_2^{-\frac{1}{2}} \frac{a_2^{-\frac{1}{2}}(1 + a_2)e(a_1, a_2) \prod_{k=1}^4 (1 - a_k a_1)(1 - a_k a_1 a_2)}{(1 - a_1)(1 - a_1 a_2)(1 - a_1/a_2)(1 - a_1 a_2^2)(1 - a_1^2 a_2)} \\ = a_1^{-2} a_2^{-1} \frac{(1 + a_2)(1 - a_1 a_2)(1 - a_1/a_2) \prod_{k=1}^4 (1 - a_k a_1)(1 - a_k a_1 a_2)}{(1 - a_1)(1 - a_1 a_2)(1 - a_1/a_2)(1 - a_1 a_2^2)(1 - a_1^2 a_2)},$$

which coincides with (6.22).  $\square$

**Proposition 6.10.** For  $e_{\varpi_2}(z)$  it follows that

$$(6.23) \quad b_2 \langle e_{\varpi_2}(z) \rangle + b_1 \langle 1 \rangle = 0,$$

where the coefficients  $b_1$  and  $b_2$  are given as

$$\begin{aligned} b_2 &= (1 + a_1 a_2 a_3 a_4), \\ b_1 &= a_1^{-1} a_2^{-1} (1 + a_1)(1 + a_2)(1 - a_1 a_2 a_3)(1 - a_1 a_2 a_4). \end{aligned}$$

**Proof.** Applying Lemma 6.6 to Lemma 6.9, we immediately have (6.23). Dividing both sides of (6.20) by  $a_1^{-1}(1 - a_1 a_2)(1 - a_1 a_3)(1 - a_1 a_4)$ , we obtain the explicit forms of the coefficients  $b_1$  and  $b_2$ .  $\square$

**Lemma 6.11.** For  $i = 0, 1$  it follows that

$$(6.24) \quad \frac{\nabla_{\text{sym}}((z_1^2 z_2^3)^{-\frac{1}{2}} + (z_1^2 z_2^3)^{\frac{1}{2}})(z_2 + z_2^{-1})^i}{4} \in \mathcal{F}_2,$$

which are expanded as

$$(6.25) \quad \frac{\nabla_{\text{sym}}((z_1^2 z_2^3)^{-\frac{1}{2}} + (z_1^2 z_2^3)^{\frac{1}{2}})}{4} = q^{-\frac{1}{2}}(E_0 - qE_4^2)s_{\varpi_1}(z) + \cdots,$$

$$(6.26) \quad \frac{\nabla_{\text{sym}}((z_1^2 z_2^3)^{-\frac{1}{2}} + (z_1^2 z_2^3)^{\frac{1}{2}})(z_2 + z_2^{-1})}{4} = q^{-\frac{1}{2}}(E_0 - qE_4^2) \left[ s_{2\varpi_2}(z) - s_{\varpi_1}(z) \right] + \cdots,$$

where  $E_r$  ( $r = 0, 1, \dots, 4$ ) is the  $r$ th elementary symmetric polynomial of  $a_1, \dots, a_4$  given by (6.18).

**Proof.** The proof follows by direct computation and we omit the details.  $\square$

**Lemma 6.12.** If we put

$$(6.27) \quad \begin{aligned} \varphi(z) &= (z_1^2 z_2^3)^{-\frac{1}{2}}(1 + z_1^2 z_2^3)e(a_1, z_2) \\ &= ((z_1^2 z_2^3)^{-\frac{1}{2}} + (z_1^2 z_2^3)^{\frac{1}{2}})((a_1 + a_1^{-1}) - (z_1 + z_2^{-1})), \end{aligned}$$

then, we have

$$(6.28) \quad \frac{\nabla_{\text{sym}}\varphi(z)}{4} = C_3 g(a_1; z) + C_2 e_{\varpi_2}(z) + C_1,$$

where the coefficients  $C_1$ ,  $C_2$  and  $C_3$  are given as

$$(6.29) \quad C_3 = q^{-\frac{1}{2}}(1 - qa_1^2 a_2^2 a_3^2 a_4^2),$$

$$(6.30) \quad C_2 = q^{-\frac{1}{2}} a_1^{-2} (1 - a_1 a_2)(1 - a_1 a_3)(1 - a_1 a_4)(1 + qa_1^3 a_2 a_3 a_4),$$

$$(6.31) \quad \begin{aligned} C_1 &= q^{-\frac{1}{2}} a_1^{-3} a_2^{-1} (1 - a_1 a_2)(1 - a_1 a_3)(1 - a_1 a_4) \\ &\quad \times (1 - a_1 a_2 a_3)(1 - a_1 a_2 a_4)(1 + a_1)(1 + qa_1^2 a_2). \end{aligned}$$

**Proof.** Applying (6.25) and (6.26) of Lemma 6.11 to (6.27) we have  $\nabla_{\text{sym}}\varphi(z)/4 \in \mathcal{F}_2$ . Then,  $\nabla_{\text{sym}}\varphi(z)/4$  can be expanded in terms of the basis of  $\mathcal{F}_2$  defined in §5.3.1, namely

$$\frac{\nabla_{\text{sym}}\varphi(z)}{4} = C_3 g(a_1; z) + C_{2.5} e_{\varpi_1}(z) + C_2 e_{\varpi_2}(z) + C_1,$$

where  $C_1$ ,  $C_2$ ,  $C_{2.5}$  and  $C_3$  are some constants independent of  $z$ .

We first evaluate  $C_3$  and show  $C_{2.5} = 0$ . Using expansion (5.13) of  $g(a_1; z)$  we have

$$(6.32) \quad \begin{aligned} \frac{\nabla_{\text{sym}}\varphi(z)}{4} &= C_3 \left( -s_{2\varpi_2}(z) + (a_1 + a_1^{-1} + 1)s_{\varpi_1}(z) + \cdots \right) \\ &\quad + C_{2.5} \left( s_{\varpi_1}(z) + \cdots \right) + C_2 e_{\varpi_2}(z) + C_1 \\ &= -C_3 s_{2\varpi_2}(z) + \left( (a_1 + a_1^{-1} + 1)C_3 + C_{2.5} \right) s_{\varpi_1}(z) + \cdots . \end{aligned}$$

On the other hand, from (6.25) and (6.26) we have an expansion

$$(6.33) \quad \frac{\nabla_{\text{sym}}\varphi(z)}{4} = -q^{-\frac{1}{2}}(1 - qa_1^2 a_2^2 a_3^2 a_4^2) \left[ s_{2\varpi_2}(z) - \left( (a_1 + a_1^{-1} + 1) \right) s_{\varpi_1}(z) \right] + \cdots .$$

Comparing coefficients of  $s_{\varpi_1}(z)$  and  $s_{2\varpi_2}(z)$  in (6.32) and (6.33), we obtain  $C_{2.5} = 0$  and  $C_3 = q^{-\frac{1}{2}}(1 - qa_1^2 a_2^2 a_3^2 a_4^2)$ .

Next we evaluate  $C_1$ . Since  $e_{\varpi_2}(\mathfrak{p}_{12}) = 0$  and  $g(a_1; \mathfrak{p}_{12}) = 0$ , from (6.28) we have  $C_1 = \nabla_{\text{sym}}\varphi(\mathfrak{p}_{12})/4$ . By Lemma 6.7 with  $F_2(\mathfrak{p}_{12}) = F_3(\mathfrak{p}_{12}) = F_4(\mathfrak{p}_{12}) = F_5(\mathfrak{p}_{12}) = 0$  and  $\varphi_1(\mathfrak{p}_{12}) = 0$ , using

$$\begin{aligned} F_0(z)\varphi_0(z) &= F_+(z)T_{q,z_1}^{\frac{1}{2}}\varphi(z) \\ &= q^{-\frac{1}{2}} \frac{(1 + qz_1^2 z_2^3)(1 - a_1 z_2)(1 - a_1/z_2) \prod_{k=1}^4 (1 - a_k z_1 z_2)(1 - a_k z_1 z_2^2)}{a_1 z_1^2 z_2^3 (1 - z_1 z_2)(1 - z_1 z_2^2)(1 - z_1)(1 - z_1 z_2^3)(1 - z_1^2 z_2^3)} , \end{aligned}$$

we obtain

$$\begin{aligned} C_1 &= \frac{\nabla_{\text{sym}}\varphi(\mathfrak{p}_{12})}{4} = F_0(\mathfrak{p}_{12})\varphi_0(\mathfrak{p}_{12}) = F_+(\mathfrak{p}_{12})T_{q,z_1}^{\frac{1}{2}}\varphi(\mathfrak{p}_{12}) \\ &= q^{-\frac{1}{2}} \frac{(1 + qa_1^2 a_2)(1 - a_1 a_2)(1 - a_1/a_2) \prod_{k=1}^4 (1 - a_k a_1)(1 - a_k a_1 a_2)}{a_1^3 a_2 (1 - a_1)(1 - a_1 a_2)(1 - a_1/a_2)(1 - a_1 a_2^2)(1 - a_1^2 a_2)} , \end{aligned}$$

which coincides with (6.31).

Lastly we evaluate  $C_2$ . Since  $g(a_1; \mathfrak{p}_{12}^*) = 0$ , from (6.28) we obtain

$$C_2 = \frac{1}{e_{\varpi_2}(\mathfrak{p}_{12}^*)} \left( \frac{\nabla_{\text{sym}}\varphi(\mathfrak{p}_{12}^*)}{4} - C_1 \right).$$

By Lemma 6.7 with  $F_1(\mathfrak{p}_{12}^*) = F_2(\mathfrak{p}_{12}^*) = F_3(\mathfrak{p}_{12}^*) = F_4(\mathfrak{p}_{12}^*) = 0$ , we obtain

$$\frac{\nabla_{\text{sym}}\varphi(\mathfrak{p}_{12}^*)}{4} = F_0(\mathfrak{p}_{12}^*)\varphi_0(\mathfrak{p}_{12}^*) + F_5(\mathfrak{p}_{12}^*)\varphi_5(\mathfrak{p}_{12}^*).$$

Thus we have

$$(6.34) \quad C_2 = \frac{1}{e_{\varpi_2}(\mathfrak{p}_{12}^*)} \left( F_0(\mathfrak{p}_{12}^*)\varphi_0(\mathfrak{p}_{12}^*) + F_5(\mathfrak{p}_{12}^*)\varphi_5(\mathfrak{p}_{12}^*) - C_1 \right).$$

From the explicit form

$$F_5(z)\varphi_5(z) = q^{-\frac{1}{2}} \frac{(1 + qz_1)(1 - a_1 z_1 z_2^2)(1 - a_1/z_1 z_2^2) \prod_{k=1}^4 (1 - a_k/z_2)(1 - a_k z_1 z_2)}{a_1 z_1 (1 - z_2^{-1})(1 - z_1 z_2)(1 - 1/z_1 z_2^3)(1 - z_1^2 z_2^3)(1 - z_1)} ,$$

we have

$$\begin{aligned}
& F_5(p_{12}^*)\varphi_5(p_{12}^*) \\
&= q^{-\frac{1}{2}} \frac{(1 + qa_1^2/a_2)(1 - a_1a_2)(1 - a_1/a_2) \prod_{k=1}^4 (1 - a_k a_1/a_2)(1 - a_k a_1)}{a_1^3 a_2^{-1} (1 - a_1/a_2)(1 - a_1)(1 - a_1/a_2^2)(1 - a_1a_2)(1 - a_1^2/a_2)} \\
(6.35) \quad &= q^{-\frac{1}{2}} (1 - a_1a_2)(1 - a_1a_3)(1 - a_1a_4) \\
&\quad \times \frac{(1 + qa_1^2/a_2)(1 - a_1^2)(1 - a_1a_3/a_2)(1 - a_1a_4/a_2)}{a_1^3 a_2^{-1} (1 - a_1/a_2^2)},
\end{aligned}$$

and in the same manner we have

$$\begin{aligned}
(6.36) \quad F_0(p_{12}^*)\varphi_0(p_{12}^*) &= -q^{-\frac{1}{2}} (1 + qa_1a_2)(1 + a_1)(1 - a_2^2)(1 - a_1a_2) \\
&\quad \times \frac{(1 - a_1a_3)(1 - a_1a_4)(1 - a_2a_3)(1 - a_2a_4)}{a_1 a_2^3 (1 - a_1/a_2^2)}.
\end{aligned}$$

Using (6.31), (6.35) and (6.36), a factor  $F_0(p_{12}^*)\varphi_0(p_{12}^*) + F_5(p_{12}^*)\varphi_5(p_{12}^*) - C_1$  in the expression (6.34) of  $C_2$  is computed as

$$\begin{aligned}
& F_0(p_{12}^*)\varphi_0(p_{12}^*) + F_5(p_{12}^*)\varphi_5(p_{12}^*) - C_1 \\
&= -q^{-\frac{1}{2}} \frac{(1 + a_1)(1 - a_1a_2)(1 - a_1a_3)(1 - a_1a_4)}{a_1^3 a_2^3 (1 - a_1/a_2^2)} \\
&\quad \times \left[ a_1^2 (1 + qa_1a_2)(1 - a_2^2)(1 - a_2a_3)(1 - a_2a_4) \right. \\
&\quad \quad \left. - a_2^4 (1 + qa_1^2/a_2)(1 - a_1)(1 - a_1a_3/a_2)(1 - a_1a_4/a_2) \right. \\
&\quad \quad \left. + a_2^2 (1 - a_1a_2a_3)(1 - a_1a_2a_4)(1 + qa_1^2a_2)(1 - a_1/a_2^2) \right] \\
&\quad \text{(the sum of these three terms is factored into a product of binomials)} \\
&= -q^{-\frac{1}{2}} \frac{(1 + a_1)(1 - a_1a_2)(1 - a_1a_3)(1 - a_1a_4)}{a_1^3 a_2^3 (1 - a_1/a_2^2)} \\
&\quad \times a_2^2 (1 - a_1)(1 - a_2^2)(1 - a_1/a_2^2)(1 + qa_1^3 a_2 a_3 a_4) \\
&= -q^{-\frac{1}{2}} a_1^{-3} a_2^{-1} (1 - a_1^2)(1 - a_2^2)(1 - a_1a_2)(1 - a_1a_3)(1 - a_1a_4)(1 + qa_1^3 a_2 a_3 a_4).
\end{aligned}$$

Since  $e_{\varpi_2}(p_{12}^*) = -a_1^{-1} a_2^{-1} (1 - a_1^2)(1 - a_2^2)$  by (5.18), using the above identity, (6.34) is computed as

$$\begin{aligned}
C_2 &= (-1) q^{-\frac{1}{2}} a_1^{-3} a_2^{-1} (1 - a_1^2)(1 - a_2^2)(1 - a_1a_2)(1 - a_1a_3)(1 - a_1a_4)(1 + qa_1^3 a_2 a_3 a_4) \\
&\quad \times \frac{-a_1 a_2}{(1 - a_1^2)(1 - a_2^2)},
\end{aligned}$$

which coincides with (6.30). □

**Proposition 6.13.** For  $g(a_1; z)$  and  $e_{\varpi_2}(z)$  it follows that

$$(6.37) \quad c_3 \langle g(a_1; z) \rangle + c_2 \langle e_{\varpi_2}(z) \rangle + c_1 \langle 1 \rangle = 0,$$

where the coefficient  $c_1$ ,  $c_2$  and  $c_3$  are given as

$$\begin{aligned}
c_3 &= (1 - qa_1^2 a_2^2 a_3^2 a_4^2), \\
c_2 &= a_1^{-2} (1 - a_1a_2)(1 - a_1a_3)(1 - a_1a_4)(1 + qa_1^3 a_2 a_3 a_4),
\end{aligned}$$

$$c_1 = a_1^{-3} a_2^{-1} (1 - a_1 a_2) (1 - a_1 a_3) (1 - a_1 a_4) \\ \times (1 - a_1 a_2 a_3) (1 - a_1 a_2 a_4) (1 + a_1) (1 + q a_1^2 a_2).$$

**Proof.** Applying Lemma 6.6 to Lemma 6.12, we immediately have (6.37). Multiplying both sides of (6.28) by  $q^{\frac{1}{2}}$ , we obtain the explicit forms of the coefficients  $c_1$ ,  $c_2$  and  $c_3$ .  $\square$

We conclude this subsection by proving Proposition 6.2.

**Proof of Proposition 6.2.** Eliminating  $\langle e_{\varpi_2}(z) \rangle$  from (6.23) and (6.37), i.e., (6.38)  $0 = b_2 c_3 \langle g(a_1; z) \rangle + b_2 c_2 \langle e_{\varpi_2}(z) \rangle + b_2 c_1 \langle 1 \rangle = b_2 c_3 \langle g(a_1; z) \rangle + (-b_1 c_2 + b_2 c_1) \langle 1 \rangle$ , we obtain

$$\langle g(a_1; z) \rangle = \frac{b_1 c_2 - b_2 c_1}{b_2 c_3} \langle 1 \rangle$$

The numerator of the coefficient is computed as

$$\begin{aligned} & b_1 c_2 - b_2 c_1 \\ &= a_1^{-1} a_2^{-1} (1 + a_1) (1 + a_2) (1 - a_1 a_2 a_3) (1 - a_1 a_2 a_4) \\ &\quad \times a_1^{-2} (1 - a_1 a_2) (1 - a_1 a_3) (1 - a_1 a_4) (1 + q a_1^3 a_2 a_3 a_4) \\ &\quad - (1 + a_1 a_2 a_3 a_4) \times a_1^{-3} a_2^{-1} (1 - a_1 a_2) (1 - a_1 a_3) (1 - a_1 a_4) \\ &\quad \times (1 - a_1 a_2 a_3) (1 - a_1 a_2 a_4) (1 + a_1) (1 + q a_1^2 a_2) \\ &= a_1^{-3} a_2^{-1} (1 + a_1) (1 - a_1 a_2) (1 - a_1 a_3) (1 - a_1 a_4) (1 - a_1 a_2 a_3) (1 - a_1 a_2 a_4) \\ &\quad \times \left[ (1 + a_2) (1 + q a_1^3 a_2 a_3 a_4) - (1 + a_1 a_2 a_3 a_4) (1 + q a_1^2 a_2) \right] \\ &= a_1^{-3} a_2^{-1} (1 + a_1) (1 - a_1 a_2) (1 - a_1 a_3) (1 - a_1 a_4) (1 - a_1 a_2 a_3) (1 - a_1 a_2 a_4) \\ &\quad \times a_2 (1 - a_1 a_3 a_4) (1 - q a_1^2) \\ &= a_1^{-3} (1 + a_1) (1 - q a_1^2) \prod_{k=2}^4 (1 - a_1 a_k) \prod_{2 \leq i < j \leq 4} (1 - a_1 a_i a_j), \end{aligned}$$

and the denominator of the coefficient is  $b_2 c_3 = (1 + a_1 a_2 a_3 a_4) (1 - q a_1^2 a_2^2 a_3^2 a_4^2)$ . Therefore we obtain

$$\langle g(a_1; z) \rangle = \frac{a_1^{-3} (1 + a_1) (1 - q a_1^2) \prod_{k=2}^4 (1 - a_1 a_k) \prod_{2 \leq i < j \leq 4} (1 - a_1 a_i a_j)}{(1 + a_1 a_2 a_3 a_4) (1 - q a_1^2 a_2^2 a_3^2 a_4^2)} \langle 1 \rangle.$$

Since we have  $I(q a_1, a_2, a_3, a_4) = \langle a_1^3 g(a_1; z) \rangle$  and  $I(a_1, a_2, a_3, a_4) = \langle 1 \rangle$  by definition of the integral  $I(a_1, a_2, a_3, a_4)$ , the above relation between  $\langle g(a_1; z) \rangle$  and  $\langle 1 \rangle$  implies the  $q$ -difference equation (6.3) in Proposition 6.2.  $\square$

#### 6.4. Special value of the integral – Proof of Lemma 6.4.

**Lemma 6.14.**

$$(6.39) \quad \frac{1}{(2\pi\sqrt{-1})^2} \iint_{\mathbb{T}^2} (z_1, z_1 z_2^3, z_1^2 z_2^3; q)_\infty (z_1^{-1}, z_1^{-1} z_2^{-3}, z_1^{-2} z_2^{-3}; q)_\infty \frac{dz_1}{z_1} \frac{dz_2}{z_2} \\ = \frac{1}{(2\pi\sqrt{-1})^2} \iint_{\mathbb{T}^2} (z_2, z_1 z_2, z_1 z_2^2; q)_\infty (z_2^{-1}, z_1^{-1} z_2^{-1}, z_1^{-1} z_2^{-2}; q)_\infty \frac{dz_1}{z_1} \frac{dz_2}{z_2}.$$

**Proof.** This is confirmed by variable change  $w = z_2^3$  for the left-hand side.  $\square$

The claim of Lemma 6.4 is that the left-hand side of (6.39) coincides with  $6/(q; q)_\infty^2$ . Instead of this it suffices to prove that the right-hand side of (6.39) is equal to  $6/(q; q)_\infty^2$ . Writing the right-hand side of (6.39) as an iterated integral, we have

$$(6.40) \quad \begin{aligned} & \frac{1}{(2\pi\sqrt{-1})^2} \iint_{\mathbb{T}^2} (z_2, z_1 z_2, z_1 z_2^2; q)_\infty (z_2^{-1}, z_1^{-1} z_2^{-1}, z_1^{-1} z_2^{-2}; q)_\infty \frac{dz_1}{z_1} \frac{dz_2}{z_2} \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z_2, z_2^{-1}; q)_\infty \\ & \quad \times \left( \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z_1 z_2, z_1^{-1} z_2^{-1}, z_1 z_2^2, z_1^{-1} z_2^{-2}; q)_\infty \frac{dz_1}{z_1} \right) \frac{dz_2}{z_2}. \end{aligned}$$

We first compute the integral of  $z_1$ -variable.

$$\begin{aligned} & \frac{(q; q)_\infty^2}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z_1 z_2, z_1^{-1} z_2^{-1}, z_1 z_2^2, z_1^{-1} z_2^{-2}; q)_\infty \frac{dz_1}{z_1} \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (1 - z_1^{-1} z_2^{-1})(1 - z_1^{-1} z_2^{-2}) z_1^{-1} \vartheta(z_1 z_2; q) \vartheta(z_1 z_2^2; q) dz_1. \end{aligned}$$

The integrand as a function of  $z_1$  has the unique essential singularity at  $z_1 = 0$ . Using the triple product (2.5) of Jacobi's theta function, the integrand is expanded as

$$\begin{aligned} & (1 - z_1^{-1} z_2^{-1})(1 - z_1^{-1} z_2^{-2}) z_1^{-1} \vartheta(z_1 z_2; q) \vartheta(z_1 z_2^2; q) \\ &= \left( z_1^{-1} - z_1^{-2}(z_2^{-1} + z_2^{-2}) + z_1^{-3} z_2^{-3} \right) \sum_{m, n=-\infty}^{\infty} (-z_1 z_2)^n q^{\binom{n}{2}} (-z_1 z_2^2)^m q^{\binom{m}{2}}, \end{aligned}$$

so that the residue (the coefficient of  $z_1^{-1}$ ) is equal to the sum of  $A, B, C$  written below.

$$\begin{aligned} A &= \sum_{n+m=0} z_2^{n+2m} q^{\binom{n}{2} + \binom{m}{2}} = \sum_{m=-\infty}^{\infty} z_2^m q^{\binom{-m}{2} + \binom{m}{2}} = \sum_{m=-\infty}^{\infty} z_2^m q^{m^2}, \\ B &= -(z_2^{-1} + z_2^{-2}) \sum_{n+m=1} (-1) z_2^{n+2m} q^{\binom{n}{2} + \binom{m}{2}} = \sum_{n+m=1} (z_2^{n+2m-1} + z_2^{n+2m-2}) q^{\binom{n}{2} + \binom{m}{2}} \\ &= \sum_{m=-\infty}^{\infty} (z_2^m + z_2^{m-1}) q^{\binom{1-m}{2} + \binom{m}{2}} = \sum_{m=-\infty}^{\infty} (z_2^m + z_2^{m-1}) q^{m(m-1)}, \\ C &= z_2^{-3} \sum_{n+m=2} z_2^{n+2m} q^{\binom{n}{2} + \binom{m}{2}} = \sum_{n+m=2} z_2^{n+2m-3} q^{\binom{n}{2} + \binom{m}{2}} = \sum_{m=-\infty}^{\infty} z_2^{m-1} q^{\binom{2-m}{2} + \binom{m}{2}} \\ &= \sum_{m=-\infty}^{\infty} z_2^{m-1} q^{(m-1)^2}. \end{aligned}$$

Therefore we obtain

$$(6.41) \quad \begin{aligned} & \frac{(q; q)_\infty^2}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z_1 z_2, z_1^{-1} z_2^{-1}, z_1 z_2^2, z_1^{-1} z_2^{-2}; q)_\infty \frac{dz_1}{z_1} \\ &= 2 \sum_{m=-\infty}^{\infty} z_2^m q^{m^2} + \sum_{m=-\infty}^{\infty} (z_2^m + z_2^{m-1}) q^{m(m-1)}. \end{aligned}$$

From (6.40), we have

$$\begin{aligned}
& \frac{(q; q)_\infty^3}{(2\pi\sqrt{-1})^2} \iint_{\mathbb{T}^2} (z_2, z_1 z_2, z_1 z_2^2; q)_\infty (z_2^{-1}, z_1^{-1} z_2^{-1}, z_1^{-1} z_2^{-2}; q)_\infty \frac{dz_1}{z_1} \frac{dz_2}{z_2} \\
&= \frac{(q; q)_\infty}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z_2, z_2^{-1}; q)_\infty \\
(6.42) \quad & \times \left( \frac{(q; q)_\infty^2}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z_1 z_2, z_1^{-1} z_2^{-1}, z_1 z_2^2, z_1^{-1} z_2^{-2}; q)_\infty \frac{dz_1}{z_1} \right) \frac{dz_2}{z_2} \\
&= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z_2^{-1} - z_2^{-2}) \vartheta(z_2; q) \\
& \times \left( \frac{(q; q)_\infty^2}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z_1 z_2, z_1^{-1} z_2^{-1}, z_1 z_2^2, z_1^{-1} z_2^{-2}; q)_\infty \frac{dz_1}{z_1} \right) dz_2.
\end{aligned}$$

From (6.41), the above integrand as a function of  $z_2$  is expanded as

$$\begin{aligned}
& (z_2^{-1} - z_2^{-2}) \sum_{n=-\infty}^{\infty} (-z_2)^n q^{\binom{n}{2}} \left( 2 \sum_{m=-\infty}^{\infty} z_2^m q^{m^2} + \sum_{m=-\infty}^{\infty} (z_2^m + z_2^{m-1}) q^{m(m-1)} \right) \\
&= \sum_{n=-\infty}^{\infty} (-z_2)^n q^{\binom{n}{2}} \left( 2 \sum_{m=-\infty}^{\infty} (z_2^{m-1} - z_2^{m-2}) q^{m^2} + \sum_{m=-\infty}^{\infty} (z_2^{m-1} - z_2^{m-3}) q^{m(m-1)} \right),
\end{aligned}$$

whose residue (the coefficient of  $z_2^{-1}$ ) is computed as

$$\begin{aligned}
& 2 \sum_{n+m=0} (-1)^n q^{\binom{n}{2}+m^2} + 2 \sum_{n+m=1} (-1)^{n+1} q^{\binom{n}{2}+m^2} \\
& + \sum_{n+m=0} (-1)^n q^{\binom{n}{2}+m(m-1)} + \sum_{n+m=2} (-1)^{n+1} q^{\binom{n}{2}+m(m-1)}.
\end{aligned}$$

Using the triple product (2.5) again, we obtain

$$\begin{aligned}
\sum_{n+m=0} (-1)^n q^{\binom{n}{2}+m^2} &= \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}+(-n)^2} = \sum_{n=-\infty}^{\infty} (-q)^n q^{3\binom{n}{2}} \\
&= \vartheta(q; q^3) = (q, q^2, q^3; q^3)_\infty = (q; q)_\infty, \\
\sum_{n+m=1} (-1)^{n+1} q^{\binom{n}{2}+m^2} &= \sum_{m=-\infty}^{\infty} (-1)^{2-m} q^{\binom{1-m}{2}+m^2} = \sum_{m=-\infty}^{\infty} (-1)^m q^{\binom{m}{2}+m^2} \\
&= (q; q)_\infty, \\
\sum_{n+m=0} (-1)^n q^{\binom{n}{2}+m(m-1)} &= \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}+n(n+1)} = \sum_{n=-\infty}^{\infty} (-q^2)^n q^{3\binom{n}{2}} \\
&= \vartheta(q^2; q^3) = (q^2, q, q^3; q^3)_\infty = (q; q)_\infty, \\
\sum_{n+m=2} (-1)^{n+1} q^{\binom{n}{2}+m(m-1)} &= \sum_{n=-\infty}^{\infty} (-1)^{n+1} q^{\binom{n}{2}+(n-1)(n-2)} \\
&= \sum_{n=-\infty}^{\infty} (-q)^{n-1} q^{3\binom{n-1}{2}} = \vartheta(q; q^3) = (q; q)_\infty.
\end{aligned}$$

Consequently, the left-hand side of (6.42) is written as

$$\frac{(q; q)_\infty^3}{(2\pi\sqrt{-1})^2} \iint_{\mathbb{T}^2} (z_2, z_1 z_2, z_1 z_2^2; q)_\infty (z_2^{-1}, z_1^{-1} z_2^{-1}, z_1^{-1} z_2^{-2}; q)_\infty \frac{dz_1}{z_1} \frac{dz_2}{z_2} = 6(q; q)_\infty,$$

which is our claim of this subsection.

## 7. NASSRALLAH–RAHMAN INTEGRAL OF TYPE $G_2$

In this section we define the Nassrallah–Rahman integral of type  $G_2$  extending the Askey–Wilson integral of type  $G_2$  (Gustafson’s integral), and discuss two  $q$ -difference equations for the Nassrallah–Rahman integral of type  $G_2$ .

**7.1. Nassrallah–Rahman  $q$ -hypergeometric integral of type  $G_2$  – Definition and Results.** Let  $\Phi(z)$  be function of  $z = (z_1, z_2) \in (\mathbb{C}^*)^2$  defined by

$$(7.1) \quad \Phi(z) = \Phi_+(z)\Phi_+(z^{-1}),$$

where  $z^{-1} = (z_1^{-1}, z_2^{-1})$  and

$$\Phi_+(z) = \frac{(z_2, z_1 z_2, z_1 z_2^2, z_1, z_1 z_2^3, z_1^2 z_2^3; q)_\infty}{\prod_{k=1}^5 (a_k z_2, a_k z_1 z_2, a_k z_1 z_2^2; q)_\infty} (qa_6^{-1} z_2, qa_6^{-1} z_1 z_2, qa_6^{-1} z_1 z_2^2; q)_\infty.$$

We define  $J(a_1, a_2, a_3, a_4, a_5, a_6)$  as

$$J(a_1, a_2, a_3, a_4, a_5, a_6) = \frac{1}{(2\pi\sqrt{-1})^2} \iint_{\mathbb{T}^2} \Phi(z) \frac{dz_1}{z_1} \frac{dz_2}{z_2}$$

One of the main result is the following.

**Theorem 7.1.** *Under the condition  $a_1 a_2 \cdots a_6 = -1$ , the integral  $J(a_1, \dots, a_6)$  satisfies*

$$(7.2) \quad \begin{aligned} & C_1 J(qa_1, a_2, a_3, a_4, a_5, a_6) + C_5 J(a_1, a_2, a_3, a_4, qa_5, a_6) \\ & + C_6 J(a_1, a_2, a_3, a_4, a_5, qa_6) = 0 \end{aligned}$$

where the coefficient  $C_1, C_5, C_6$  is given as

$$\begin{aligned} C_1 &= a_1^2 (1 - a_1) (a_5^2 - a_6^2) \prod_{k=1}^6 \frac{1}{1 - a_k a_1} \prod_{k=2}^4 (1 - a_k a_5 a_6), \\ C_5 &= a_5^2 (1 - a_5) (a_6^2 - a_1^2) \prod_{k=1}^6 \frac{1}{1 - a_k a_5} \prod_{k=2}^4 (1 - a_k a_1 a_6), \\ C_6 &= a_6^8 (1 - a_6) (a_1^2 - a_5^2) \prod_{k=1}^6 \frac{1}{1 - a_k a_6} \prod_{k=2}^4 (1 - a_k a_1 a_5). \end{aligned}$$

**Proof.** The proof will be given in §7.3. □

The other result of this paper is the following:

**Theorem 7.2.** *Under the condition  $a_1 a_2 \cdots a_6 q = -1$ , the integral  $J(a_1, \dots, a_6)$  satisfies*

$$(7.3) \quad \begin{aligned} & C_{56} J(a_1, a_2, a_3, a_4, qa_5, qa_6) + C_{16} J(qa_1, a_2, a_3, a_4, a_5, qa_6) \\ & + C_{15} J(qa_1, a_2, a_3, a_4, qa_5, a_6) = 0 \end{aligned}$$



where the coefficient  $C_{56}$ ,  $C_{16}$ ,  $C_{15}$  is given as

$$\begin{aligned}
C_{56} &= a_1 \frac{(1-a_1^2)(1-qa_1^2)}{(1-a_1)} \frac{(1-qa_1a_5)(1-qa_1a_6)}{(1-a_1^2/a_5^2)(1-a_1^2/a_6^2)} \\
&\quad \times \prod_{k=2}^4 (1-a_1a_k) \prod_{2 \leq i < j \leq 4} (1-a_1a_i a_j), \\
C_{16} &= a_5 \frac{(1-a_5^2)(1-qa_5^2)}{(1-a_5)} \frac{(1-qa_5a_1)(1-qa_5a_6)}{(1-a_5^2/a_1^2)(1-a_5^2/a_6^2)} \\
&\quad \times \prod_{k=2}^4 (1-a_5a_k) \prod_{2 \leq i < j \leq 4} (1-a_5a_i a_j), \\
C_{15} &= \frac{(1-a_6^2)(1-qa_6^2)}{a_6^5(1-a_6)} \frac{(1-qa_6a_1)(1-qa_6a_5)}{(1-a_6^2/a_1^2)(1-a_6^2/a_5^2)} \\
&\quad \times \prod_{k=2}^4 (1-a_6a_k) \prod_{2 \leq i < j \leq 4} (1-a_6a_i a_j).
\end{aligned}$$

**Proof.** The proof will be given in §7.4.  $\square$

**Remark.** As a corollary of Theorem 7.2, when we consider the limit  $a_5 \rightarrow 0$  ( $a_6^{-1} \rightarrow 0$ ), the  $q$ -difference equation (6.3) in Proposition 6.2 for Gustafson integral  $I(a_1, a_2, a_3, a_4)$  can be obtained from (7.3). Here we confirm this fact. Under the condition  $a_1 a_2 \cdots a_6 q = -1$ , if  $a_5 \rightarrow 0$ , then we have the limits

$$\begin{aligned}
J(a_1, a_2, a_3, a_4, qa_5, qa_6) &= J(a_1, a_2, a_3, a_4, qa_5, -a_1^{-1} \cdots a_5^{-1}) \rightarrow I(a_1, a_2, a_3, a_4), \\
J(qa_1, a_2, a_3, a_4, a_5, qa_6) &= J(qa_1, a_2, a_3, a_4, qa_5, -a_1^{-1} \cdots a_5^{-1}) \rightarrow I(qa_1, a_2, a_3, a_4), \\
J(qa_1, a_2, a_3, a_4, qa_5, a_6) &= J(qa_1, a_2, a_3, a_4, qa_5, -q^{-1} a_1^{-1} \cdots a_5^{-1}) \rightarrow I(qa_1, a_2, a_3, a_4),
\end{aligned}$$

so that we have

$$a_6 C_{56} I(a_1, a_2, a_3, a_4) + a_6 (C_{16} + C_{15}) I(qa_1, a_2, a_3, a_4) = 0.$$

where the coefficients  $a_6 C_{56}$ ,  $a_6 C_{16}$ ,  $a_6 C_{15}$  are computed as follows.

$$\begin{aligned}
a_6 C_{56} &= a_1 a_6 \frac{(1-a_1^2)(1-qa_1^2)}{(1-a_1)} \frac{(1-qa_1a_5)(1-qa_1a_6)}{(1-a_1^2/a_5^2)(1-a_1^2/a_6^2)} \\
&\quad \times \prod_{k=2}^4 (1-a_1a_k) \prod_{2 \leq i < j \leq 4} (1-a_1a_i a_j) \\
&= qa_5^2 a_6^2 \frac{(1-a_1^2)(1-qa_1^2)}{(1-a_1)} \frac{(1-qa_1a_5)(1-1/qa_1a_6)}{(1-a_5^2/a_1^2)(1-a_1^2/a_6^2)} \\
&\quad \times \prod_{k=2}^4 (1-a_1a_k) \prod_{2 \leq i < j \leq 4} (1-a_1a_i a_j) \\
&\rightarrow qa_5^2 a_6^2 \frac{(1-a_1^2)(1-qa_1^2)}{(1-a_1)} \prod_{k=2}^4 (1-a_1a_k) \prod_{2 \leq i < j \leq 4} (1-a_1a_i a_j)
\end{aligned}$$

$$\begin{aligned}
&= \frac{(1+a_1)(1-qa_1^2)}{qa_1^2a_2^2a_3^2a_4^2} \prod_{k=2}^4 (1-a_1a_k) \prod_{2 \leq i < j \leq 4} (1-a_1a_i a_j), \\
a_6C_{16} &= a_5a_6 \frac{(1-a_5^2)(1-qa_5^2)}{(1-a_5)} \frac{(1-qa_5a_1)(1-qa_5a_6)}{(1-a_5^2/a_1^2)(1-a_5^2/a_6^2)} \\
&\quad \times \prod_{k=2}^4 (1-a_5a_k) \prod_{2 \leq i < j \leq 4} (1-a_5a_i a_j) \\
&= -qa_5^2a_6^2 \frac{(1-a_5^2)(1-qa_5^2)}{(1-a_5)} \frac{(1-qa_5a_1)(1-1/qa_5a_6)}{(1-a_5^2/a_1^2)(1-a_5^2/a_6^2)} \\
&\quad \times \prod_{k=2}^4 (1-a_5a_k) \prod_{2 \leq i < j \leq 4} (1-a_5a_i a_j) \\
&\rightarrow -qa_5^2a_6^2(1-1/qa_5a_6) = -\frac{(1+a_1a_2a_3a_4)}{qa_1^2a_2^2a_3^2a_4^2}, \\
a_6C_{15} &= \frac{(1-a_6^2)(1-qa_6^2)}{a_6^4(1-a_6)} \frac{(1-qa_6a_1)(1-qa_6a_5)}{(1-a_6^2/a_1^2)(1-a_6^2/a_5^2)} \\
&\quad \times \prod_{k=2}^4 (1-a_6a_k) \prod_{2 \leq i < j \leq 4} (1-a_6a_i a_j) \\
&= \frac{q^3a_1a_5a_6^6}{-a_6^9/a_1^2a_5^2} \frac{(1-1/a_6^2)(1-1/qa_6^2)}{(1-1/a_6)} \frac{(1-1/qa_6a_1)(1-1/qa_6a_5)}{(1-a_1^2/a_6^2)(1-a_5^2/a_6^2)} \\
&\quad \times a_6^6a_2^3a_3^3a_4^3 \prod_{k=2}^4 (1-1/a_6a_k) \prod_{2 \leq i < j \leq 4} (1-1/a_6a_i a_j) \\
&= -q^3a_1^3a_2^3a_3^3a_4^3a_5^3a_6^3 \frac{(1-1/a_6^2)(1-1/qa_6^2)}{(1-1/a_6)} \frac{(1-1/qa_6a_1)(1-1/qa_6a_5)}{(1-a_1^2/a_6^2)(1-a_5^2/a_6^2)} \\
&\quad \times \prod_{k=2}^4 (1-1/a_6a_k) \prod_{2 \leq i < j \leq 4} (1-1/a_6a_i a_j) \\
&= \frac{(1-1/a_6^2)(1-1/qa_6^2)}{(1-1/a_6)} \frac{(1-1/qa_6a_1)(1-1/qa_6a_5)}{(1-a_1^2/a_6^2)(1-a_5^2/a_6^2)} \\
&\quad \times \prod_{k=2}^4 (1-1/a_6a_k) \prod_{2 \leq i < j \leq 4} (1-1/a_6a_i a_j) \\
&\rightarrow (1-1/qa_5a_6) = (1+a_1a_2a_3a_4).
\end{aligned}$$

Therefore, if  $a_5 \rightarrow 0$ , consequently we have

$$\begin{aligned}
a_6C_{56} &\rightarrow \frac{(1+a_1)(1-qa_1^2)}{qa_1^2a_2^2a_3^2a_4^2} \prod_{k=2}^4 (1-a_1a_k) \prod_{2 \leq i < j \leq 4} (1-a_1a_i a_j), \\
a_6C_{16} &\rightarrow -\frac{(1+a_1a_2a_3a_4)}{qa_1^2a_2^2a_3^2a_4^2}, \\
a_6C_{15} &\rightarrow (1+a_1a_2a_3a_4),
\end{aligned}$$

$$\begin{aligned} a_6(C_{16} + C_{15}) &\rightarrow (1 - qa_5^2 a_6^2)(1 - 1/qa_5 a_6) = -qa_5^2 a_6^2(1 - 1/qa_5^2 a_6^2)(1 - 1/qa_5 a_6) \\ &= -\frac{(1 + a_1 a_2 a_3 a_4)(1 - qa_1^2 a_2^2 a_3^2 a_4^2)}{qa_1^2 a_2^2 a_3^2 a_4^2}. \end{aligned}$$

Therefore we had the following from

$$\frac{I(qa_1, a_2, a_3, a_4)}{I(a_1, a_2, a_3, a_4)} = \frac{(1 + a_1)(1 - qa_1^2)}{(1 + a_1 a_2 a_3 a_4)(1 - qa_1^2 a_2^2 a_3^2 a_4^2)} \prod_{i=2}^4 (1 - a_1 a_i) \prod_{2 \leq j < k \leq 4} (1 - a_1 a_j a_k),$$

which coincides with the  $q$ -difference equation (6.3) in Proposition 6.2.  $\square$

Under the condition  $a_1 a_2 \cdots a_6 = -q$ , we want to find the system of  $q$ -difference equations for  $J(a_1, a_2, a_3, a_4, a_5, a_6)$ . Since  $a_6$  is written as  $a_6 = -q(a_1 a_2 \cdots a_5)^{-1}$   $J(a_1, a_2, a_3, a_4, a_5, a_6)$  is equal to

$$\bar{J} = \bar{J}(a_1, a_2, a_3, a_4, a_5) := J(a_1, a_2, a_3, a_4, a_5, -q(a_1 a_2 \cdots a_5)^{-1}),$$

which we regard  $\bar{J}$  as a function of  $(a_1, a_2, a_3, a_4, a_5)$ . If we consider the result in Theorem 7.1 under  $a_6 \rightarrow q^{-1} a_6$ , and that in Theorem 7.2 under  $a_6 \rightarrow q^{-2} a_6$ , then we immediately have the following:

**Corollary 7.3.** *Under the condition  $a_1 a_2 \cdots a_6 = -q$ , the integral  $\bar{J}(a_1, a_2, a_3, a_4, a_5)$  satisfies the system of first-order simultaneous  $q$ -difference equations of rank 2 as*

$$T_{q, a_1}(\bar{J}, T_{q, a_5} \bar{J}) = (\bar{J}, T_{q, a_5} \bar{J})A,$$

where the  $2 \times 2$  matrix  $A$  is given in terms of Gauss matrix decomposition as

$$A = \begin{pmatrix} \alpha & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} 1 & \gamma \\ 0 & \delta \end{pmatrix}^{-1}$$

Here the entries  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$  are given as  $\alpha = -T_{q, a_6}^{-1}(C_6/C_1)$ ,  $\beta = -T_{q, a_6}^{-1}(C_5/C_1)$ ,  $\gamma = -T_{q^2, a_6}^{-1}(C_{16}/C_{56})$  and  $\delta = -T_{q^2, a_6}^{-1}(C_{15}/C_{56})$ , respectively.

**Remark 1.** The determinant of the coefficient matrix  $A$  is given as

$$\det A = \alpha/\delta = \frac{T_{q, a_6}^{-1}(C_6/C_1)}{T_{q^2, a_6}^{-1}(C_{15}/C_{56})}.$$

**Remark 2.** Eliminating  $T_{q, a_5} \bar{J}$  from this system, the second order  $q$ -difference equation for  $\bar{J}$  can be obtained.

## 7.2. $q$ -Stokes formula.

**Lemma 7.4.** *Let  $F_+(z)$  and  $F_-(z)$  be functions specified as*

$$(7.4) \quad F_+(z) = (z_1^2 z_2^3)^{-\frac{3}{2}} \frac{\prod_{k=1}^6 (1 - a_k z_1 z_2)(1 - a_k z_1 z_2^2)}{(1 - z_1 z_2)(1 - z_1 z_2^2)(1 - z_1)(1 - z_1 z_2^3)(1 - z_1^2 z_2^3)}.$$

Then, it follows that

$$\frac{T_{q, z_1} \Phi(z)}{\Phi(z)} = -\frac{F_+(z)}{T_{q, z_1} F_-(z)}, \quad F_-(z) = F_+(z^{-1}).$$

**Proof.** By the definition (7.1) of  $\Phi(z)$  we have

$$\begin{aligned}
\frac{T_{q,z_1}\Phi(z)}{\Phi(z)} &= \frac{(1 - (qz_1z_2)^{-1})(1 - (qz_1z_2^2)^{-1})(1 - (qz_1)^{-1})}{(1 - z_1z_2)(1 - z_1z_2^2)(1 - z_1)} \\
&\quad \times \frac{(1 - (qz_1z_2^3)^{-1})(1 - (qz_1^2z_2^3)^{-1})(1 - (q^2z_1^2z_2^3)^{-1})}{(1 - z_1z_2^3)(1 - z_1^2z_2^3)(1 - qz_1^2z_2^3)} \\
&\quad \times \frac{(1 - qa_6^{-1}(qz_1z_2)^{-1})(1 - qa_6^{-1}(qz_1z_2^2)^{-1})}{(1 - qa_6^{-1}z_1z_2)(1 - qa_6^{-1}z_1z_2^2)} \\
&\quad \times \prod_{k=1}^5 \frac{(1 - a_kz_1z_2)(1 - a_kz_1z_2^2)}{(1 - a_k(qz_1z_2)^{-1})(1 - a_k(qz_1z_2^2)^{-1})} \\
&= -(qz_1^2z_2^3)^{-3} \frac{(1 - (qz_1z_2)^{-1})(1 - (qz_1z_2^2)^{-1})}{(1 - z_1z_2)(1 - z_1z_2^2)} \\
&\quad \times \frac{(1 - (qz_1)^{-1})(1 - (qz_1z_2^3)^{-1})(1 - (q^2z_1^2z_2^3)^{-1})}{(1 - z_1)(1 - z_1z_2^3)(1 - z_1^2z_2^3)} \\
&\quad \times \prod_{k=1}^6 \frac{(1 - a_kz_1z_2)(1 - a_kz_1z_2^2)}{(1 - a_k(qz_1z_2)^{-1})(1 - a_k(qz_1z_2^2)^{-1})}
\end{aligned}$$

This implies Lemma 7.4.  $\square$

We remark that if we change the definitions of  $\Phi(z)$  and  $F_+(z)$  in §6.2 to (7.1) and (7.4) in this section, respectively, then the same arguments as  $q$ -Stokes' theorem (Lemma 6.6) and Lemma 6.7 hold.

**7.3.  $q$ -Difference equation – Proof of Theorem 7.1.** From the definition (7.1) of  $\Phi(z)$ , we have

$$\frac{T_{q,a_k}\Phi(z)}{\Phi(z)} = a_k^3 g(a_k; z) \quad (k = 1, 2, 3, 4, 5), \quad \frac{T_{q,a_6}\Phi(z)}{\Phi(z)} = a_6^{-3} g(a_6; z),$$

so that

$$\begin{aligned}
J(qa_1, a_2, a_3, a_4, a_5, a_6) &= a_1^3 \langle g(a_1; z) \rangle, \\
J(a_1, a_2, a_3, a_4, qa_5, a_6) &= a_5^3 \langle g(a_5; z) \rangle, \\
J(a_1, a_2, a_3, a_4, a_5, qa_6) &= a_6^{-3} \langle g(a_6; z) \rangle.
\end{aligned}$$

Taking account of symmetry of parameters, Theorem 7.1 is equivalent to the following:

**Proposition 7.5.** *Under the condition  $a_1a_2 \cdots a_6 = -1$ , the integral  $J(a_1, \dots, a_6)$  satisfies*

$$(7.5) \quad C_1 \langle g(a_1; z) \rangle + C_2 \langle g(a_2; z) \rangle + C_3 \langle g(a_3; z) \rangle = 0,$$

where

$$(7.6) \quad C_1 = \frac{a_1^5(1 - a_1)(a_2^2 - a_3^2)}{(1 - a_2^2a_3)(1 - a_2a_3^2)} \prod_{k=1}^6 \frac{1 - a_k a_2 a_3}{1 - a_k a_1},$$

$$(7.7) \quad C_2 = \frac{a_2^5(1 - a_2)(a_3^2 - a_1^2)}{(1 - a_1^2a_3)(1 - a_1a_3^2)} \prod_{k=1}^6 \frac{1 - a_k a_1 a_3}{1 - a_k a_2},$$

$$(7.8) \quad C_3 = \frac{a_3^5(1-a_3)(a_1^2-a_2^2)}{(1-a_1^2a_2)(1-a_1a_2^2)} \prod_{k=1}^6 \frac{1-a_k a_1 a_2}{1-a_k a_3}.$$

The aim of this subsection is to prove Proposition 7.5. Before proving it we show two lemmas.

**Lemma 7.6.** *For  $i = 0, 1, 2$  it follows that*

$$(7.9) \quad \frac{\nabla_{\text{sym}}(z_2 + z_2^{-1})^i (z_2^{-\frac{1}{2}} + z_2^{\frac{1}{2}})}{4} \in \mathcal{F}_3$$

*The explicit expressions of (7.9) are given by*

$$(7.10) \quad \frac{\nabla_{\text{sym}}(z_2^{-\frac{1}{2}} + z_2^{\frac{1}{2}})}{4} = (S_0 + S_6)(S_0 - S_6)s_{\varpi_1 + \varpi_2}(z) + \cdots,$$

$$(7.11) \quad \frac{\nabla_{\text{sym}}(z_2 + z_2^{-1})(z_2^{-\frac{1}{2}} + z_2^{\frac{1}{2}})}{4} \\ = (S_0 + S_6)(S_0 - S_6)s_{3\varpi_2}(z) + 0s_{\varpi_1 + \varpi_2}(z) + \cdots,$$

$$(7.12) \quad \frac{\nabla_{\text{sym}}(z_2 + z_2^{-1})^2 (z_2^{-\frac{1}{2}} + z_2^{\frac{1}{2}})}{4} \\ = (S_0 + S_6) \left[ (S_1 - S_5)s_{3\varpi_2}(z) + (S_4 - S_2 + S_0 - S_6) \right] s_{\varpi_1 + \varpi_2}(z) + \cdots,$$

where  $S_r$  ( $r = 0, 1, \dots, 6$ ) is the  $r$ th elementary symmetric polynomial of  $a_1, \dots, a_6$  given by

$$(7.13) \quad S_r = \sum_{1 \leq i_1 < \cdots < i_r \leq 6} a_{i_1} a_{i_2} \cdots a_{i_r}$$

i.e.,  $S_0 = 1$ ,  $S_1 = a_1 + \cdots + a_6$ ,  $\dots$ ,  $S_6 = a_1 \cdots a_6$ . In particular, under the condition

$$a_1 a_2 \cdots a_6 = -1$$

it follows that

$$(7.14) \quad \frac{\nabla_{\text{sym}}(z_2 + z_2^{-1})^i (z_2^{-\frac{1}{2}} + z_2^{\frac{1}{2}})}{4} \in \mathcal{F}_2 \quad (i = 0, 1, 2).$$

**Proof.** We can confirm (7.10)–(7.12) by direct calculation. Since all of the coefficients of  $s_{3\varpi_2}(z)$  and  $s_{\varpi_1 + \varpi_2}(z)$  in the right-hand sides of (7.10)–(7.12) have the factor  $S_0 + S_6$ , under the condition  $a_1 a_2 \cdots a_6 = -1$ , i.e.,  $S_0 + S_6 = 0$ , we obtain (7.14).  $\square$

**Lemma 7.7.** *Suppose that  $a_1 a_2 \cdots a_6 = -1$ . Put*

$$\begin{aligned} \varphi(z) &= z_2^{-\frac{1}{2}}(1+z_2)e(a_1; z_2)e(a_2; z_2) \\ &= z_2^{-\frac{1}{2}}(1+z_2)(1-a_1 z_2)(1-a_1 z_2^{-1})(1-a_2 z_2)(1-a_2 z_2^{-1})a_1^{-1}a_2^{-1} \end{aligned}$$

Then,  $\nabla_{\text{sym}}\varphi(z)/4$  is expressed as

$$(7.15) \quad \frac{\nabla_{\text{sym}}\varphi(z)}{4} = c_1 g(a_1; z) + c_2 g(a_2; z) + c_{12} G(z) \in \mathcal{F}_2,$$

where the coefficients  $c_1$ ,  $c_2$ ,  $c_{12}$  are given as

$$(7.16) \quad c_1 = a_1^2 a_2^{-4} a_3^{-2} \frac{(1+a_2)(1+a_3)(1-a_2 a_3)}{(1-a_1/a_2)(1-a_1/a_2 a_3)} \prod_{k=4}^6 (1-a_k a_2)(1-a_k a_2 a_3),$$

$$(7.17) \quad c_2 = a_1^{-4} a_2^2 a_3^{-2} \frac{(1+a_1)(1+a_3)(1-a_1 a_3)}{(1-a_2/a_1)(1-a_2/a_1 a_3)} \prod_{k=4}^6 (1-a_k a_1)(1-a_k a_1 a_3),$$

$$(7.18) \quad c_{12} = a_1^{-2} a_2^{-3} (1+a_2/a_1)(1-a_1 a_2)(1-a_1 a_2 a_3)(1-a_1 a_2/a_3) \\ \times \prod_{k=3}^6 (1-a_k a_1)(1-a_k a_2).$$

**Proof.** We set  $\tilde{\varphi}(z) = \nabla_{\text{sym}} \varphi(z)/4$ . Since  $\varphi(z)$  is expanded as

$$\varphi(z) = (z_2^{-\frac{1}{2}} + z_2^{\frac{1}{2}}) \left[ (z_2 + z_2^{-1})^2 - (A_1 + A_2)(z_2 + z_2^{-1}) + A_1 A_2 + 2 \right], \quad A_i = a_i + a_i^{-1}$$

From (7.14) of Lemma 7.7, under the condition  $a_1 a_2 \cdots a_6 = -1$ , we have  $\tilde{\varphi}(z) \in \mathcal{F}_2$ , namely we have the expansion as

$$\tilde{\varphi}(z) = c_1 g(a_1; z) + c_2 g(a_2; z) + c_3 g(a_3; z) + c_{12} G(z),$$

where  $c_1, c_2, c_3$  and  $c_{12}$  are the constants independent of  $z$ . From the vanishing properties of  $g(a_i, z)$  and  $G(z)$ , we immediately have

$$c_1 = \frac{\tilde{\varphi}(\mathbf{p}_{23})}{g(a_1; \mathbf{p}_{23})}, \quad c_2 = \frac{\tilde{\varphi}(\mathbf{p}_{13})}{g(a_2; \mathbf{p}_{13})}, \quad c_3 = \frac{\tilde{\varphi}(\mathbf{p}_{12})}{g(a_3; \mathbf{p}_{12})}, \quad c_{12} = \frac{\tilde{\varphi}(\mathbf{p}_{12}^*) - c_3 g(a_3; \mathbf{p}_{12}^*)}{G(\mathbf{p}_{12}^*)}$$

We first evaluate  $c_1, c_2, c_3$ . From Lemma 6.7  $\tilde{\varphi}(z)$  is written as

$$\tilde{\varphi}(z) = \frac{\nabla_{\text{sym}} \varphi(z)}{4} = \sum_{k=0}^5 F_k(z) \varphi_k(z).$$

If  $z = \mathbf{p}_{ij}$  (i.e.,  $z_2 = a_j$ ,  $z_1 z_2 = a_i$ ,  $z_1 z_2^2 = a_i a_j$ ), then  $F_2(\mathbf{p}_{ij}) = F_3(\mathbf{p}_{ij}) = F_4(\mathbf{p}_{ij}) = F_5(\mathbf{p}_{ij}) = 0$ , so that we have

$$\tilde{\varphi}(\mathbf{p}_{ij}) = F_0(\mathbf{p}_{ij}) \varphi_0(\mathbf{p}_{ij}) + F_1(\mathbf{p}_{ij}) \varphi_1(\mathbf{p}_{ij}).$$

Since  $\varphi(z)$  does not include  $z_1$ , we have

$$\varphi_0(z) = \varphi(z) = z_2^{-\frac{1}{2}} (1+z_2) e(a_1; z_2) e(a_2; z_2), \\ \varphi_1(z) = (z_1 z_2)^{-\frac{1}{2}} (1+z_1 z_2) e(a_1; z_1 z_2) e(a_2; z_1 z_2),$$

so that we have

$$\varphi_0(\mathbf{p}_{ij}) = a_j^{-\frac{1}{2}} (1+a_j) e(a_1; a_j) e(a_2; a_j), \\ \varphi_1(\mathbf{p}_{ij}) = a_i^{-\frac{1}{2}} (1+a_i) e(a_1; a_i) e(a_2; a_i).$$

In particular  $\varphi_1(\mathbf{p}_{1j}) = 0$  and

$$\varphi_0(\mathbf{p}_{1j}) = a_j^{-\frac{1}{2}} (1+a_j) e(a_1; a_j) e(a_2; a_j) \\ = a_j^{-\frac{1}{2}} (1+a_j) \frac{(1-a_1 a_j)(1-a_1/a_j)(1-a_2 a_j)(1-a_2/a_j)}{a_1 a_2}.$$

Therefore we obtain

$$\tilde{\varphi}(\mathbf{p}_{1j}) = F_0(\mathbf{p}_{1j}) \varphi_0(\mathbf{p}_{1j}) = F_+(\mathbf{p}_{1j}) \varphi(\mathbf{p}_{1j}).$$

Since  $\varphi(\mathbf{p}_{12}) = 0$ , we obtain

$$c_3 = \frac{\tilde{\varphi}(\mathbf{p}_{12})}{g(a_3; \mathbf{p}_{12})} = \frac{F_+(\mathbf{p}_{12}) \varphi(\mathbf{p}_{12})}{g(a_3; \mathbf{p}_{12})} = 0.$$

Moreover, we can calculate

$$\begin{aligned}
c_2 &= \frac{\tilde{\varphi}(\mathbf{p}_{13})}{g(a_2; \mathbf{p}_{13})} = \frac{F_+(\mathbf{p}_{13})\varphi(\mathbf{p}_{13})}{g(a_2; \mathbf{p}_{13})} \\
&= (a_1^2 a_3)^{-\frac{3}{2}} \frac{\prod_{k=1}^6 (1 - a_k a_1)(1 - a_k a_1 a_3)}{(1 - a_1)(1 - a_1 a_3)(1 - a_1/a_3)(1 - a_1 a_3^2)(1 - a_1^2 a_3)} \\
&\quad \times a_3^{-\frac{1}{2}} (1 + a_3) \frac{(1 - a_1 a_3)(1 - a_1/a_3)(1 - a_2 a_3)(1 - a_2/a_3)}{a_1 a_2} \\
&\quad \times \frac{a_3^3}{(1 - a_2 a_1)(1 - a_2/a_1)(1 - a_2 a_3)(1 - a_2/a_3)(1 - a_1 a_2 a_3)(1 - a_2/a_1 a_3)} \\
&= a_1^{-4} a_2^2 a_3^{-2} \frac{(1 + a_1)(1 + a_3)(1 - a_1 a_3)}{(1 - a_2/a_1)(1 - a_2/a_1 a_3)} \prod_{k=4}^6 (1 - a_k a_1)(1 - a_k a_1 a_3),
\end{aligned}$$

which is (7.17). From symmetry of the equations  $c_1$  is obtained from  $c_2$  changing  $a_1 \leftrightarrow a_2$ . Thus we have

$$c_1 = a_1^2 a_2^{-4} a_3^{-2} \frac{(1 + a_2)(1 + a_3)(1 - a_2 a_3)}{(1 - a_1/a_2)(1 - a_1/a_2 a_3)} \prod_{k=4}^6 (1 - a_k a_2)(1 - a_k a_2 a_3),$$

which coincides with (7.16).

We next evaluate  $c_{12}$ . From Lemma 6.7, if  $z = \mathbf{p}_{ij}^*$  (i.e.,  $z_2 = a_j/a_i$ ,  $z_1 z_2 = a_i$ ,  $z_1 z_2^2 = a_j$ ), then  $F_2(\mathbf{p}_{ij}^*) = F_3(\mathbf{p}_{ij}^*) = F_4(\mathbf{p}_{ij}^*) = 0$ , so that we have

$$\tilde{\varphi}(\mathbf{p}_{ij}^*) = F_0(\mathbf{p}_{ij}^*)\varphi_0(\mathbf{p}_{ij}^*) + F_1(\mathbf{p}_{ij}^*)\varphi_1(\mathbf{p}_{ij}^*) + F_5(\mathbf{p}_{ij}^*)\varphi_5(\mathbf{p}_{ij}^*).$$

Since  $\varphi(z)$  does not include  $z_1$ , we have

$$\begin{aligned}
\varphi_0(z) &= \varphi(z) = z_2^{-\frac{1}{2}} (1 + z_2) e(a_1; z_2) e(a_2; z_2), \\
\varphi_1(z) &= (z_1 z_2)^{-\frac{1}{2}} (1 + z_1 z_2) e(a_1; z_1 z_2) e(a_2; z_1 z_2), \\
\varphi_5(z) &= (z_1 z_2^2)^{-\frac{1}{2}} (1 + z_1 z_2^2) e(a_1; z_1 z_2^2) e(a_2; z_1 z_2^2),
\end{aligned}$$

so that we have

$$\begin{aligned}
\varphi_0(\mathbf{p}_{ij}^*) &= (a_j/a_i)^{-\frac{1}{2}} (1 + a_j/a_i) e(a_1; a_j/a_i) e(a_2; a_j/a_i), \\
\varphi_1(\mathbf{p}_{ij}^*) &= a_i^{-\frac{1}{2}} (1 + a_i) e(a_1; a_i) e(a_2; a_i), \\
\varphi_5(\mathbf{p}_{ij}^*) &= a_j^{-\frac{1}{2}} (1 + a_j) e(a_1; a_j) e(a_2; a_j).
\end{aligned}$$

From  $\varphi_1(\mathbf{p}_{12}^*) = \varphi_5(\mathbf{p}_{12}^*) = 0$ , we have

$$\begin{aligned}
\varphi_0(\mathbf{p}_{12}^*) &= (a_2/a_1)^{-\frac{1}{2}} (1 + a_2/a_1) e(a_1; a_2/a_1) e(a_2; a_2/a_1) \\
&= (a_2/a_1)^{-\frac{1}{2}} (1 + a_2/a_1)(1 - a_2)(1 - a_1^2/a_2)(1 - a_2^2/a_1)(1 - a_1) a_1^{-1} a_2^{-1},
\end{aligned}$$

so that we have

$$\begin{aligned}
\tilde{\varphi}(\mathbf{p}_{12}^*) &= F_0(\mathbf{p}_{12}^*)\varphi_0(\mathbf{p}_{12}^*) = F_+(\mathbf{p}_{12}^*)\varphi(\mathbf{p}_{12}^*) \\
&= (a_1 a_2)^{-\frac{3}{2}} \frac{\prod_{k=1}^6 (1 - a_k a_1)(1 - a_k a_2)}{(1 - a_1)(1 - a_2)(1 - a_1^2/a_2)(1 - a_2^2/a_1)(1 - a_1 a_2)} \\
&\quad \times (a_2/a_1)^{-\frac{1}{2}} (1 + a_2/a_1)(1 - a_2)(1 - a_1^2/a_2)(1 - a_2^2/a_1)(1 - a_1) a_1^{-1} a_2^{-1}
\end{aligned}$$

$$= \frac{(1 + a_2/a_1) \prod_{k=1}^6 (1 - a_k a_1)(1 - a_k a_2)}{a_1^2 a_2^3 (1 - a_1 a_2)}.$$

Using the fact that  $c_3 = 0$ , we obtain

$$\begin{aligned} c_{12} &= \frac{\tilde{\varphi}(p_{12}^*) - c_3 g(a_3; p_{12}^*)}{G(p_{12}^*)} = \frac{\tilde{\varphi}(p_{12}^*)}{G(p_{12}^*)} \\ &= \frac{(1 + a_2/a_1) \prod_{k=1}^6 (1 - a_k a_1)(1 - a_k a_2)}{a_1^2 a_2^3 (1 - a_1 a_2)} \times \frac{a_1 a_2 e(a_1 a_2, a_3)}{(1 - a_1^2)(1 - a_2^2)} \\ &= a_1^{-2} a_2^{-3} (1 + a_2/a_1)(1 - a_1 a_2)(1 - a_1 a_2 a_3)(1 - a_1 a_2/a_3) \\ &\quad \times \prod_{k=3}^6 (1 - a_k a_1)(1 - a_k a_2), \end{aligned}$$

which coincides with (7.18).  $\square$

**Proof of Proposition 7.5.** Using Lemma 6.6, (7.15) of Lemma 7.7 implies

$$(7.19) \quad c_1 \langle g(a_1; z) \rangle + c_2 \langle g(a_2; z) \rangle + c_{12} \langle G(z) \rangle = 0,$$

where  $c_1, c_2, c_{12}$  are given as (7.16)–(7.18). If we consider (7.15) of Lemma 7.7 with  $a_2 \leftrightarrow a_3$ , we obtain

$$(7.20) \quad d_1 \langle g(a_1; z) \rangle + d_3 \langle g(a_3; z) \rangle + d_{13} \langle G(z) \rangle = 0$$

where,  $d_1, d_3, d_{13}$  are given as (7.16)–(7.18) with  $a_2 \leftrightarrow a_3$ , i.e.,

$$\begin{aligned} d_1 &= a_1^2 a_3^{-4} a_2^{-2} \frac{(1 + a_2)(1 + a_3)(1 - a_2 a_3)}{(1 - a_1/a_3)(1 - a_1/a_2 a_3)} \prod_{k=4}^6 (1 - a_k a_3)(1 - a_k a_2 a_3), \\ d_3 &= a_1^{-4} a_3^2 a_2^{-2} \frac{(1 + a_1)(1 + a_2)(1 - a_1 a_2)}{(1 - a_3/a_1)(1 - a_3/a_1 a_2)} \prod_{k=4}^6 (1 - a_k a_1)(1 - a_k a_1 a_2), \\ d_{13} &= a_1^{-2} a_3^{-3} (1 + a_3/a_1)(1 - a_1 a_2)(1 - a_1 a_3)(1 - a_2 a_3)(1 - a_1 a_2 a_3)(1 - a_1 a_3/a_2) \\ &\quad \times \prod_{k=4}^6 (1 - a_k a_1)(1 - a_k a_3). \end{aligned}$$

Eliminating  $\langle G(z) \rangle$  from (7.19) and (7.20), we have the relation (7.5).  $\square$

**7.4.  $q$ -Difference equation – Proof of Theorem 7.2.** By definition we have

$$\begin{aligned} J(a_1, a_2, a_3, a_4, qa_5, qa_6) &= a_5^3 a_6^{-3} \langle g(a_5; z) g(a_6; z) \rangle, \\ J(qa_1, a_2, a_3, a_4, qa_5, a_6) &= a_1^3 a_5^3 \langle g(a_1; z) g(a_5; z) \rangle, \\ J(qa_1, a_2, a_3, a_4, a_5, qa_6) &= a_1^3 a_6^{-3} \langle g(a_1; z) g(a_6; z) \rangle. \end{aligned}$$

Taking account of symmetry of the parameters, Theorem 7.2 is equivalent to the following.

**Proposition 7.8.** *Under the condition  $a_1 a_2 \cdots a_6 q = -1$ , it follows that*

$$(7.21) \quad C_{23} \langle g(a_2; z) g(a_3; z) \rangle + C_{13} \langle g(a_1; z) g(a_3; z) \rangle + C_{12} \langle g(a_1; z) g(a_2; z) \rangle = 0,$$



where

$$C_{23} = \frac{(1-a_1^2)(1-qa_1^2)}{a_1^2(1-a_1)} \frac{(1-qa_1a_2)(1-qa_1a_3)}{(1-a_1^2/a_2^2)(1-a_1^2/a_3^2)} \prod_{k=4}^6 (1-a_1a_k) \prod_{4 \leq i < j \leq 6} (1-a_1a_i a_j),$$

$$C_{13} = \frac{(1-a_2^2)(1-qa_2^2)}{a_2^2(1-a_2)} \frac{(1-qa_2a_1)(1-qa_2a_3)}{(1-a_2^2/a_1^2)(1-a_2^2/a_3^2)} \prod_{k=4}^6 (1-a_2a_k) \prod_{4 \leq i < j \leq 6} (1-a_2a_i a_j),$$

$$C_{12} = \frac{(1-a_3^2)(1-qa_3^2)}{a_3^2(1-a_3)} \frac{(1-qa_3a_1)(1-qa_3a_2)}{(1-a_3^2/a_1^2)(1-a_3^2/a_2^2)} \prod_{k=4}^6 (1-a_3a_k) \prod_{4 \leq i < j \leq 6} (1-a_3a_i a_j).$$

The aim of this subsection is to prove Proposition 7.8. Before proving it we show four lemmas.

**Lemma 7.9.** *Let  $\alpha_i(z)$ ,  $\beta_i(z)$ ,  $\gamma_i(z)$  be functions specified by*

$$\alpha_i(z) := \frac{1}{4} \nabla_{\text{sym}} \left( (z_2 + z_2^{-1})^i (z_2^{-\frac{1}{2}} + z_2^{\frac{1}{2}}) \right),$$

$$\beta_i(z) := \frac{1}{4} \nabla_{\text{sym}} \left( (z_2 + z_2^{-1})^i (z_2^{-\frac{1}{2}} + z_2^{\frac{1}{2}}) ((z_1 z_2)^{-1} + z_1 z_2 + (z_1 z_2^2)^{-1} + z_1 z_2^2) \right),$$

$$\gamma_i(z) := \frac{1}{4} \nabla_{\text{sym}} \left( (z_2 + z_2^{-1})^i (z_2^{-\frac{1}{2}} + z_2^{\frac{1}{2}}) ((z_1 z_2)^{-1} + z_1 z_2) ((z_1 z_2^2)^{-1} + z_1 z_2^2) \right).$$

Then, we have

$$(7.22) \quad \alpha_i(z) \in \mathcal{F}_4 \quad (i = 0, 1, 2, 3), \quad \beta_i(z) \in \mathcal{F}_4 \quad (i = 0, 1, 2)$$

and

$$\beta_3(z) \in \mathcal{F}_5, \quad \gamma_i(z) \in \mathcal{F}_5 \quad (i = 0, 1, 2, 3).$$

$\beta_3(z) \in \mathcal{F}_5$  and  $\gamma_i(z) \in \mathcal{F}_5$  ( $i = 0, 1, 2, 3$ ) are expressed as

$$\beta_3(z) = q^{-\frac{1}{2}} (S_0 + qS_6) \times \left[ (S_6 - S_0) s_{5\varpi_2}(z) + (S_1 - S_5) s_{\varpi_1+3\varpi_2}(z) + (S_4 - S_2) s_{2\varpi_1+\varpi_2}(z) \right] + \cdots$$

and

$$\begin{aligned} \gamma_0(z) &= q^{-1} (S_0 + qS_6) (S_0 - qS_6) s_{2\varpi_1+\varpi_2}(z) + \cdots, \\ \gamma_1(z) &= q^{-1} (S_0 + qS_6) (S_0 - qS_6) \left[ s_{\varpi_1+3\varpi_2}(z) + 0 s_{2\varpi_1+\varpi_2}(z) \right] + \cdots, \\ \gamma_2(z) &= q^{-1} (S_0 + qS_6) (S_0 - qS_6) \left[ s_{5\varpi_2}(z) + 0 s_{\varpi_1+3\varpi_2}(z) + s_{2\varpi_1+\varpi_2}(z) \right] + \cdots, \\ \gamma_3(z) &= q^{-1} (S_0 + qS_6) \left[ (S_1 - qS_5) s_{5\varpi_2}(z) \right. \\ &\quad \left. + (qS_4 - S_2 + 2(S_0 - qS_6) + (S_6 - qS_0)) s_{\varpi_1+3\varpi_2}(z) \right. \\ &\quad \left. + ((1-q)S_3 + qS_1 - S_5) s_{2\varpi_1+\varpi_2}(z) \right] + \cdots, \end{aligned}$$

where  $S_r$  ( $r = 0, 1, \dots, 6$ ) is the  $r$ th elementary symmetric polynomial of  $a_1, \dots, a_6$  given by (7.13). In particular, under the condition

$$a_1 a_2 \cdots a_6 q = -1$$

we have

$$(7.23) \quad \beta_3(z) \in \mathcal{F}_4, \quad \gamma_i(z) \in \mathcal{F}_4 \quad (i = 0, 1, 2, 3).$$

**Proof.** The proof follows by direct computation and we omit the details.  $\square$

**Lemma 7.10.** *Suppose that  $a_1 a_2 \cdots a_6 q = -1$ . Let  $\varphi_i(z)$  ( $i = 1, 2, 3$ ) be functions specified by*

$$\begin{aligned} \varphi_i(z) &= z_2^{-\frac{1}{2}}(1+z_2)e(a_1, z_2)e(a_2, z_2)e(a_3, z_2)e(c_i, z_1 z_2)e(c_i, z_1 z_2^2) \\ &= z_2^{-\frac{1}{2}}(1+z_2)(1-a_1 z_2)(1-a_1 z_2^{-1})(1-a_2 z_2) \\ &\quad \times (1-a_2 z_2^{-1})(1-a_3 z_2)(1-a_3 z_2^{-1})a_1^{-1}a_2^{-1}a_3^{-1} \\ &\quad \times (1-c_i z_1 z_2)(1-c_i/z_1 z_2)(1-c_i z_1 z_2^2)(1-c_i/z_1 z_2^2)c_i^{-2}, \end{aligned}$$

and  $g_{ij}(z)$ ,  $G_i(z)$  be the functions defined as (5.26), (5.28), respectively. If  $c_i = qa_i$  ( $i = 1, 2, 3$ ), then  $\nabla_{\text{sym}}\varphi_i(z)/4$  are expanded as

$$\begin{aligned} \frac{\nabla_{\text{sym}}\varphi_1(z)}{4} &= c_{13}^{(1)}g_{13}(z) + c_{12}^{(1)}g_{12}(z) + c_1^{(1)}G_1(z), \\ \frac{\nabla_{\text{sym}}\varphi_2(z)}{4} &= c_{23}^{(2)}g_{23}(z) + c_{12}^{(2)}g_{12}(z) + c_2^{(2)}G_2(z), \\ \frac{\nabla_{\text{sym}}\varphi_3(z)}{4} &= c_{23}^{(3)}g_{23}(z) + c_{13}^{(3)}g_{13}(z) + c_3^{(3)}G_3(z), \end{aligned}$$

where the coefficients are given as

$$(7.24) \quad c_1^{(1)} = q^{-1}a_2^{-2}a_3^{-2}(a_2+a_3)(1-a_2a_3)(1-qa_1a_2)(1-qa_1a_3)e(a_2a_3, a_4) \\ \times \prod_{k=4}^6(1-a_k a_2)(1-a_k a_3),$$

$$c_2^{(2)} = q^{-1}a_1^{-2}a_3^{-2}(a_1+a_3)(1-a_1a_3)(1-qa_1a_2)(1-qa_2a_3)e(a_1a_3, a_4) \\ \times \prod_{k=4}^6(1-a_k a_1)(1-a_k a_3),$$

$$c_3^{(3)} = q^{-1}a_1^{-2}a_2^{-2}(a_1+a_2)(1-a_1a_2)(1-qa_1a_3)(1-qa_2a_3)e(a_1a_2, a_4) \\ \times \prod_{k=4}^6(1-a_k a_1)(1-a_k a_2),$$

$$(7.25) \quad c_{12}^{(i)} = c_{21}^{(i)} = \frac{(1+a_3)(1+a_4)(1-a_3a_4a_5)(1-a_3a_4a_6)\prod_{k=4}^6(1-a_k a_3)}{(1-a_3a_4/a_1)(1-a_3a_4/a_2)(1-a_3/a_1)(1-a_3/a_2)} \\ \times e(q^{\frac{1}{2}}a_i, q^{\frac{1}{2}}a_3)e(q^{\frac{1}{2}}a_i, q^{\frac{1}{2}}a_3a_4),$$

$$c_{13}^{(i)} = c_{31}^{(i)} = \frac{(1+a_2)(1+a_4)(1-a_2a_4a_5)(1-a_2a_4a_6)\prod_{k=4}^6(1-a_k a_2)}{(1-a_2a_4/a_1)(1-a_2a_4/a_3)(1-a_2/a_1)(1-a_2/a_3)} \\ \times e(q^{\frac{1}{2}}a_i, q^{\frac{1}{2}}a_2)e(q^{\frac{1}{2}}a_i, q^{\frac{1}{2}}a_2a_4),$$

$$c_{23}^{(i)} = c_{32}^{(i)} = \frac{(1+a_1)(1+a_4)(1-a_1a_4a_5)(1-a_1a_4a_6)\prod_{k=4}^6(1-a_k a_1)}{(1-a_1a_4/a_2)(1-a_1a_4/a_3)(1-a_1/a_2)(1-a_1/a_3)}$$

$$\times e(q^{\frac{1}{2}}a_i, q^{\frac{1}{2}}a_1)e(q^{\frac{1}{2}}a_i, q^{\frac{1}{2}}a_1a_4).$$

**Proof.** We set  $\tilde{\varphi}_i(z) = \nabla_{\text{sym}}\varphi_i(z)/4$ . From expansion of  $\varphi_i(z)$ ,  $\tilde{\varphi}_i(z)$  can be expressed as a linear combination of  $\alpha_i(z)$ ,  $\beta_i(z)$ ,  $\gamma_i(z)$  ( $i = 0, 1, 2, 3$ ) in Lemma 7.9. This implies  $\tilde{\varphi}_i(z) \in \mathcal{F}_5$ . Under the condition  $a_1a_2 \cdots a_6q = -1$ , from (7.22) and (7.23) we have  $\tilde{\varphi}_i(z) \in \mathcal{F}_4$ . By Lemma 5.6, using basis  $\{g_{ij}(z) \mid 1 \leq i < j \leq 4\} \cup \{G_1(z), G_2(z), G_3(z)\}$  of  $\mathcal{F}_4$ ,  $\tilde{\varphi}_i(z)$  can be expanded as

$$(7.26) \quad \tilde{\varphi}_i(z) = \sum_{1 \leq j < k \leq 4} c_{jk}^{(i)} g_{jk}(z) + \sum_{k=1}^3 c_k^{(i)} G_k(z).$$

We first show  $c_{j4}^{(i)} = 0$  ( $j = 1, 2, 3$ ) and evaluate  $c_{jk}^{(i)}$  ( $1 \leq j < k \leq 3$ ). From the symmetry of parameters, without loss of generality, it suffices to show  $c_{14}^{(i)} = 0$  and the explicit form of  $c_{12}^{(i)}$ .

We consider (7.26) with  $z = p_{23}$ . From (5.29) we see  $c_{14}^{(i)} g_{14}(p_{23}) = \tilde{\varphi}_i(p_{23})$ . By Lemma 6.7 and  $F_2(p_{23}) = F_3(p_{23}) = F_4(p_{23}) = F_5(p_{23}) = 0$  and  $(\varphi_i)_0(p_{23}) = (\varphi_i)_1(p_{23}) = 0$ , we have  $\tilde{\varphi}_i(p_{23}) = 0$ . Thus we obtain

$$c_{14}^{(i)} = \frac{\tilde{\varphi}_i(p_{23})}{g_{14}(p_{23})} = 0.$$

We next consider (7.26) with  $z = p_{34}$ . From (5.29) we have  $c_{12}^{(i)} g_{12}(p_{34}) = \tilde{\varphi}_i(p_{34})$ . On the other hand By Lemma 6.7 and  $F_2(p_{34}) = F_3(p_{34}) = F_4(p_{34}) = F_5(p_{34}) = 0$  and  $(\varphi_i)_1(p_{34}) = 0$ , we obtain  $\tilde{\varphi}_i(p_{34}) = F_0(p_{34})(\varphi_i)_0(p_{34}) = F_+(p_{34})\varphi_i(q^{\frac{1}{2}}a_3/a_4, a_4)$ . Thus we can compute

$$(7.27) \quad \begin{aligned} c_{12}^{(i)} &= \frac{F_+(p_{34})\varphi_i(q^{\frac{1}{2}}a_3/a_4, a_4)}{g_{12}(p_{34})} \\ &= (a_3^2 a_4)^{-\frac{3}{2}} \frac{\prod_{k=1}^6 (1 - a_k a_3)(1 - a_k a_3 a_4)}{(1 - a_3)(1 - a_3 a_4)(1 - a_3/a_4)(1 - a_3 a_4^2)(1 - a_3^2 a_4)} \\ &\quad \times \frac{a_4^{-\frac{1}{2}}(1 + a_4)e(a_1, a_4)e(a_2, a_4)e(a_3, a_4)e(c_i, q^{\frac{1}{2}}a_3)e(c_i, q^{\frac{1}{2}}a_3 a_4)}{e(a_1, a_3)e(a_1, a_4)e(a_1, a_3 a_4)e(a_2, a_3)e(a_2, a_4)e(a_2, a_3 a_4)} \\ &= (a_3^2 a_4)^{-\frac{3}{2}} \frac{\prod_{k=1}^6 (1 - a_k a_3)(1 - a_k a_3 a_4)}{(1 - a_3)a_3 e(a_3, a_4)(1 - a_3 a_4^2)(1 - a_3^2 a_4)} \\ &\quad \times \frac{a_4^{-\frac{1}{2}}(1 + a_4)e(a_1, a_4)e(a_2, a_4)e(a_3, a_4)e(c_i, q^{\frac{1}{2}}a_3)e(c_i, q^{\frac{1}{2}}a_3 a_4)}{e(a_1, a_3)e(a_1, a_4)e(a_1, a_3 a_4)e(a_2, a_3)e(a_2, a_4)e(a_2, a_3 a_4)} \\ &= a_3^{-4} a_4^{-2} \frac{(1 + a_3)(1 + a_4)(1 - a_3 a_4 a_5)(1 - a_3 a_4 a_6) \prod_{k=4}^6 (1 - a_k a_3)}{(1 - a_3 a_4/a_1)(1 - a_3 a_4/a_2)(1 - a_3/a_1)(1 - a_3/a_2)a_3^{-4} a_4^{-2}} \\ &\quad \times e(c_i, q^{\frac{1}{2}}a_3)e(c_i, q^{\frac{1}{2}}a_3 a_4) \\ &= \frac{(1 + a_3)(1 + a_4)(1 - a_3 a_4 a_5)(1 - a_3 a_4 a_6) \prod_{k=4}^6 (1 - a_k a_3)}{(1 - a_3 a_4/a_1)(1 - a_3 a_4/a_2)(1 - a_3/a_1)(1 - a_3/a_2)} \\ &\quad \times e(q^{\frac{1}{2}}a_i, q^{\frac{1}{2}}a_3)e(q^{\frac{1}{2}}a_i, q^{\frac{1}{2}}a_3 a_4), \end{aligned}$$

which coincides with (7.25).

We next evaluate  $c_k^{(i)}$  ( $k = 1, 2, 3$ ). From the symmetry of parameters, without loss of generality, it suffices to compute  $c_1^{(i)}$ . We consider (7.26) with  $z = p_{23}^*$ . From (5.29) we have

$$(7.28) \quad \tilde{\varphi}_i(p_{23}^*) = c_1^{(i)} G_1(p_{23}^*) + c_{14}^{(i)} g_{14}(p_{23}^*) = c_1^{(i)} G_1(p_{23}^*).$$

(We have already shown  $c_{14}^{(i)} = 0$ .) On the other hand, by Lemma 6.7 and  $F_2(p_{23}^*) = F_3(p_{23}^*) = F_4(p_{23}^*) = 0$  and  $(\varphi_i)_1(p_{23}^*) = (\varphi_i)_5(p_{23}^*) = 0$ , we have

$$(7.29) \quad \tilde{\varphi}_i(p_{23}^*) = F_0(p_{23}^*)(\varphi_i)_0(p_{23}^*) = F_+(p_{23}^*)\varphi_i(q^{\frac{1}{2}}a_2^2/a_3, a_3/a_2).$$

Comparing (7.28) with (7.29),  $c_1^{(i)}$  is computed as

$$(7.30) \quad c_1^{(i)} = \frac{F_+(p_{23}^*)\varphi_i(q^{\frac{1}{2}}a_2^2/a_3, a_3/a_2)}{G_1(p_{23}^*)}$$

$$= (a_2a_3)^{-\frac{3}{2}} \frac{\prod_{k=1}^6 (1 - a_k a_2)(1 - a_k a_3)}{(1 - a_2)(1 - a_3)(1 - a_2^2/a_3)(1 - a_3^2/a_2)(1 - a_2 a_3)}$$

$$\times (a_3/a_2)^{-\frac{1}{2}} (1 + a_3/a_2) e(a_1, a_3/a_2) e(a_2, a_3/a_2) e(a_3, a_3/a_2)$$

$$\times e(c_i, q^{\frac{1}{2}}a_2) e(c_i, q^{\frac{1}{2}}a_3)$$

$$\times \frac{a_2 a_3 e(a_2 a_3, a_4)}{(1 - a_2^2)(1 - a_3^2) e(a_1, a_3/a_2) e(a_1, a_2) e(a_1, a_3)}$$

$$= (a_2a_3)^{-\frac{3}{2}} \frac{\prod_{k=1}^6 (1 - a_k a_2)(1 - a_k a_3)}{a_2 a_3 e(a_2, a_3/a_2) e(a_3, a_3/a_2) (1 - a_2 a_3)}$$

$$\times (a_3/a_2)^{-\frac{1}{2}} (1 + a_3/a_2) e(a_2, a_3/a_2) e(a_3, a_3/a_2)$$

$$\times e(c_i, q^{\frac{1}{2}}a_2) e(c_i, q^{\frac{1}{2}}a_3)$$

$$\times \frac{a_2 a_3 e(a_2 a_3, a_4)}{(1 - a_2^2)(1 - a_3^2) e(a_1, a_2) e(a_1, a_3)}$$

$$= a_2^{-1} a_3^{-2} (1 - a_1 a_2)(1 - a_1 a_3)(1 - a_2 a_3) \prod_{k=4}^6 (1 - a_k a_2)(1 - a_k a_3)$$

$$\times (1 + a_3/a_2) e(q^{\frac{1}{2}}a_i, q^{\frac{1}{2}}a_2) e(q^{\frac{1}{2}}a_i, q^{\frac{1}{2}}a_3) \frac{e(a_2 a_3, a_4)}{e(a_1, a_2) e(a_1, a_3)}.$$

Therefore, we have  $c_1^{(2)} = c_1^{(3)} = 0$  and  $c_1^{(1)}$  is evaluated as

$$c_1^{(1)} = a_2^{-1} a_3^{-2} (1 - a_1 a_2)(1 - a_1 a_3)(1 - a_2 a_3) \prod_{k=4}^6 (1 - a_k a_2)(1 - a_k a_3)$$

$$\times \frac{(1 + a_3/a_2) q^{-1} a_2^{-1} a_3^{-1} (1 - q a_1 a_2)(1 - a_2/a_1)(1 - q a_1 a_3)(1 - a_3/a_1) e(a_2 a_3, a_4)}{a_2^{-1} a_3^{-1} (1 - a_1 a_2)(1 - a_2/a_1)(1 - a_1 a_3)(1 - a_3/a_1)}$$

$$= q^{-1} a_2^{-1} a_3^{-2} e(a_2 a_3, a_4) (1 + a_3/a_2)(1 - a_2 a_3)(1 - q a_1 a_2)(1 - q a_1 a_3)$$

$$\times \prod_{k=4}^6 (1 - a_k a_2)(1 - a_k a_3),$$

which coincides with (7.24).  $\square$

**Remark.** Applying Lemma 6.6 to Lemma 7.10, we have

$$\begin{pmatrix} c_1^{(1)} & 0 & 0 \\ 0 & c_2^{(2)} & 0 \\ 0 & 0 & c_3^{(3)} \end{pmatrix} \begin{pmatrix} \langle G_1(z) \rangle \\ \langle G_2(z) \rangle \\ \langle G_3(z) \rangle \end{pmatrix} = - \begin{pmatrix} 0 & c_{13}^{(1)} & c_{12}^{(1)} \\ c_{23}^{(2)} & 0 & c_{12}^{(2)} \\ c_{23}^{(3)} & c_{13}^{(3)} & 0 \end{pmatrix} \begin{pmatrix} \langle g_{23}(z) \rangle \\ \langle g_{13}(z) \rangle \\ \langle g_{12}(z) \rangle \end{pmatrix}.$$

In other words,  $\langle G_i(z) \rangle$  can be expressed as a linear combination of  $\langle g_{12}(z) \rangle$ ,  $\langle g_{13}(z) \rangle$  and  $\langle g_{23}(z) \rangle$ , i.e.,

$$(7.31) \quad \begin{pmatrix} \langle G_1(z) \rangle \\ \langle G_2(z) \rangle \\ \langle G_3(z) \rangle \end{pmatrix} = - \begin{pmatrix} 0 & c_{13}^{(1)}/c_1^{(1)} & c_{12}^{(1)}/c_1^{(1)} \\ c_{23}^{(2)}/c_2^{(2)} & 0 & c_{12}^{(2)}/c_2^{(2)} \\ c_{23}^{(3)}/c_3^{(3)} & c_{13}^{(3)}/c_3^{(3)} & 0 \end{pmatrix} \begin{pmatrix} \langle g_{23}(z) \rangle \\ \langle g_{13}(z) \rangle \\ \langle g_{12}(z) \rangle \end{pmatrix}.$$

**Lemma 7.11.** Let  $h_i(z)$  be function specified by

$$h_i(z) := \frac{1}{4} \nabla_{\text{sym}} \left( (z_2 + z_2^{-1})^i ((z_1 z_2)^{-\frac{1}{2}} + (z_1 z_2)^{\frac{1}{2}}) ((z_1 z_2^2)^{-\frac{1}{2}} + (z_1 z_2^2)^{\frac{1}{2}}) \right).$$

If  $i = 0, 1, 2, 3$ , then  $h_i(z) \in \mathcal{F}_4$ . The explicit expressions of  $h_i(z)$  ( $i = 0, 1, 2, 3$ ) are given as

$$\begin{aligned} h_0(z) &= q^{-\frac{1}{2}} (S_0 + q^{\frac{1}{2}} S_6) (S_0 - q^{\frac{1}{2}} S_6) s_{2\varpi_1}(z) + \cdots \in \mathcal{F}_4, \\ h_1(z) &= q^{-\frac{1}{2}} (S_0 + q^{\frac{1}{2}} S_6) (S_0 - q^{\frac{1}{2}} S_6) \left[ s_{\varpi_1+2\varpi_2}(z) - s_{2\varpi_1}(z) \right] + \cdots \in \mathcal{F}_4, \\ h_2(z) &= q^{-\frac{1}{2}} (S_0 + q^{\frac{1}{2}} S_6) (S_0 - q^{\frac{1}{2}} S_6) \left[ s_{4\varpi_2}(z) - s_{\varpi_1+2\varpi_2}(z) + 2s_{2\varpi_1}(z) \right] + \cdots \in \mathcal{F}_4, \\ h_3(z) &= q^{-\frac{1}{2}} (S_0 + q^{\frac{1}{2}} S_6) \left[ \left( q^{\frac{1}{2}} (S_6 - S_5 - S_0) + (S_6 + S_1 - S_0) \right) s_{4\varpi_2}(z) \right. \\ &\quad \left. - \left( q^{\frac{1}{2}} (3S_6 - S_4 - S_1) + (S_5 + S_2 - 3S_0) \right) s_{\varpi_1+2\varpi_2}(z) \right. \\ &\quad \left. + \left( q^{\frac{1}{2}} (3S_6 - S_3 - S_2) + (S_4 + S_3 - 3S_0) \right) s_{2\varpi_1}(z) \right] + \cdots \in \mathcal{F}_4, \end{aligned}$$

where  $S_r$  ( $r = 0, 1, \dots, 6$ ) is the  $r$ th elementary symmetric polynomial of  $a_1, \dots, a_6$  given by (7.13).

**Proof.** The proof follows by direct computation and we omit the details.  $\square$

**Lemma 7.12.** Let  $\phi(z)$  be function specified by

$$\begin{aligned} \phi(z) &= (z_1^2 z_2^3)^{-\frac{1}{2}} (1 + z_1 z_2) (1 + z_1 z_2^2) e(a_1, z_2) e(a_2, z_2) e(a_3, z_2) \\ &= (z_1^2 z_2^3)^{-\frac{1}{2}} (1 + z_1 z_2) (1 + z_1 z_2^2) \\ &\quad \times (1 - a_1 z_2) (1 - a_1 z_2^{-1}) (1 - a_2 z_2) \\ &\quad \times (1 - a_2 z_2^{-1}) (1 - a_3 z_2) (1 - a_3 z_2^{-1}) a_1^{-1} a_2^{-1} a_3^{-1}, \end{aligned}$$

and  $g_{ij}(z)$ ,  $G_i(z)$  be functions defined by (5.26), (5.28), respectively. Then  $\nabla_{\text{sym}} \phi(z)/4$  is expanded as

$$(7.32) \quad \frac{\nabla_{\text{sym}} \phi(z)}{4} = d_{23} g_{23}(z) + d_{13} g_{13}(z) + d_{12} g_{12}(z) + d_1 G_1(z) + d_2 G_2(z) + d_3 G_3(z),$$

where the coefficients are given as

$$\begin{aligned}
d_{23} &= q^{-\frac{1}{2}} \frac{(1 + q^{\frac{1}{2}} a_1)(1 + q^{\frac{1}{2}} a_1 a_4)(1 + a_1)(1 - a_1 a_4 a_5)(1 - a_1 a_4 a_6)}{a_1(1 - a_1/a_2)(1 - a_1/a_3)(1 - a_1 a_4/a_2)(1 - a_1 a_4/a_3)} \\
&\quad \times \prod_{k=4}^6 (1 - a_k a_1), \\
d_{13} &= q^{-\frac{1}{2}} \frac{(1 + q^{\frac{1}{2}} a_2)(1 + q^{\frac{1}{2}} a_2 a_4)(1 + a_2)(1 - a_2 a_4 a_5)(1 - a_2 a_4 a_6)}{a_2(1 - a_2/a_1)(1 - a_2/a_3)(1 - a_2 a_4/a_1)(1 - a_2 a_4/a_3)} \\
&\quad \times \prod_{k=4}^6 (1 - a_k a_2), \\
(7.33) \quad d_{12} &= q^{-\frac{1}{2}} \frac{(1 + q^{\frac{1}{2}} a_3)(1 + q^{\frac{1}{2}} a_3 a_4)(1 + a_3)(1 - a_3 a_4 a_5)(1 - a_3 a_4 a_6)}{a_3(1 - a_3/a_1)(1 - a_3/a_2)(1 - a_3 a_4/a_1)(1 - a_3 a_4/a_2)} \\
&\quad \times \prod_{k=4}^6 (1 - a_k a_3).
\end{aligned}$$

$$\begin{aligned}
(7.34) \quad d_1 &= q^{-\frac{1}{2}} \frac{(1 + q^{\frac{1}{2}} a_2)(1 + q^{\frac{1}{2}} a_3)(1 - a_2 a_3)(1 - a_2 a_3 a_4)(1 - a_2 a_3/a_4)}{a_2^2 a_3^2 (1 - a_2/a_1)(1 - a_3/a_1)} \\
&\quad \times \prod_{k=4}^6 (1 - a_k a_2)(1 - a_k a_3), \\
d_2 &= q^{-\frac{1}{2}} \frac{(1 + q^{\frac{1}{2}} a_1)(1 + q^{\frac{1}{2}} a_3)(1 - a_1 a_3)(1 - a_1 a_3 a_4)(1 - a_1 a_3/a_4)}{a_1^2 a_3^2 (1 - a_1/a_2)(1 - a_3/a_2)} \\
&\quad \times \prod_{k=4}^6 (1 - a_k a_1)(1 - a_k a_3), \\
d_3 &= q^{-\frac{1}{2}} \frac{(1 + q^{\frac{1}{2}} a_1)(1 + q^{\frac{1}{2}} a_2)(1 - a_1 a_2)(1 - a_1 a_2 a_4)(1 - a_1 a_2/a_4)}{a_1^2 a_2^2 (1 - a_1/a_3)(1 - a_2/a_3)} \\
&\quad \times \prod_{k=4}^6 (1 - a_k a_1)(1 - a_k a_2).
\end{aligned}$$

**Remark.** This lemma does not need the balancing condition like  $a_1 \cdots a_6 = -1$  in Lemma 7.7 or  $a_1 \cdots a_6 q = -1$  in Lemma 7.10.

**Proof.** We set  $\tilde{\phi}(z) = \nabla_{\text{sym}} \phi(z)/4$ . From expansion of  $\phi(z)$   $\tilde{\phi}(z)$  can be expressed as a linear combination of  $h_i(z)$  ( $i = 0, 1, 2, 3$ ) in Lemma 7.11. This implies  $\tilde{\phi}(z) \in \mathcal{F}_4$ . By Lemma 5.6, using basis  $\{g_{ij}(z) \mid 1 \leq i < j \leq 4\} \cup \{G_1(z), G_2(z), G_3(z)\}$  of  $\mathcal{F}_4$ ,  $\tilde{\varphi}_i(z)$  can be expanded as

$$(7.35) \quad \tilde{\phi}_i(z) = \sum_{1 \leq j < k \leq 4} d_{jk} g_{jk}(z) + \sum_{k=1}^3 d_k G_k(z)$$

We first show that  $d_{j4} = 0$  ( $j = 1, 2, 3$ ) and that  $d_{jk}$  ( $1 \leq j < k \leq 3$ ) is given as (7.33). From symmetry of parameters, without loss of generality, it suffices to show  $d_{14} = 0$  and the explicit form of  $d_{12}$ .

We consider (7.35) with  $z = p_{23}$ . From (5.29) we see that  $d_{14}g_{14}(p_{23}) = \tilde{\phi}(p_{23})$ . By Lemma 6.7 and  $F_2(p_{23}) = F_3(p_{23}) = F_4(p_{23}) = F_5(p_{23}) = 0$  and  $\phi_0(p_{23}) = \phi_1(p_{23}) = 0$ , we have  $\tilde{\phi}(p_{23}) = 0$ . Thus we obtain  $d_{14} = \tilde{\phi}(p_{23})/g_{14}(p_{23}) = 0$ .

We next consider (7.35) with  $z = p_{34}$ . From (5.29), we have  $d_{12}g_{12}(p_{34}) = \tilde{\phi}(p_{34})$ . On the other hand, By Lemma 6.7 and  $F_2(p_{34}) = F_3(p_{34}) = F_4(p_{34}) = F_5(p_{34}) = 0$  and  $\phi_1(p_{34}) = 0$ , we have  $\tilde{\phi}(p_{34}) = F_0(p_{34})\phi_0(p_{34}) = F_+(p_{34})\phi(q^{\frac{1}{2}}a_3/a_4, a_4)$ . Thus we can compute

$$(7.36) \quad d_{12} = \frac{F_+(p_{34})\phi(q^{\frac{1}{2}}a_3/a_4, a_4)}{g_{12}(p_{34})},$$

which coincides with (7.33).

We next evaluate  $d_k$  ( $k = 1, 2, 3$ ). From symmetry of parameters, without loss of generality, it suffices to compute  $d_1$ . We consider (7.35) with  $z = p_{23}^*$ . From (5.29) we have

$$(7.37) \quad \tilde{\phi}(p_{23}^*) = d_1G_1(p_{23}^*) + d_{14}g_{14}(p_{23}^*) = d_1G_1(p_{23}^*).$$

(We have already shown  $d_{14} = 0$ .) On the other hand, by Lemma 6.7 and  $F_2(p_{23}^*) = F_3(p_{23}^*) = F_4(p_{23}^*) = 0$  and  $\phi_1(p_{23}^*) = \phi_5(p_{23}^*) = 0$ , we have

$$(7.38) \quad \tilde{\phi}(p_{23}^*) = F_0(p_{23}^*)\phi_0(p_{23}^*) = F_+(p_{23}^*)\phi(q^{\frac{1}{2}}a_2^2/a_3, a_3/a_2).$$

Comparing (7.37) with (7.38),  $d_1$  is computed as

$$(7.39) \quad d_1 = \frac{F_+(p_{23}^*)\phi(q^{\frac{1}{2}}a_2^2/a_3, a_3/a_2)}{G_1(p_{23}^*)},$$

which coincides with (7.34).  $\square$

**Proof of Proposition 7.8.** Applying Lemma 6.6 to Lemma 7.12, we have

$$(d_{23}, d_{13}, d_{12}) \begin{pmatrix} \langle g_{23}(z) \rangle \\ \langle g_{13}(z) \rangle \\ \langle g_{12}(z) \rangle \end{pmatrix} + (d_1, d_2, d_3) \begin{pmatrix} \langle G_1(z) \rangle \\ \langle G_2(z) \rangle \\ \langle G_3(z) \rangle \end{pmatrix} = 0.$$

Combining this with (7.31), we can eliminate  $\langle G_i(z) \rangle$  ( $i = 1, 2, 3$ ), namely

$$\left[ (d_{23}, d_{13}, d_{12}) - (d_1, d_2, d_3) \begin{pmatrix} 0 & c_{13}^{(1)}/c_1^{(1)} & c_{12}^{(1)}/c_1^{(1)} \\ c_{23}^{(2)}/c_2^{(2)} & 0 & c_{12}^{(2)}/c_2^{(2)} \\ c_{23}^{(3)}/c_3^{(3)} & c_{13}^{(3)}/c_3^{(3)} & 0 \end{pmatrix} \right] \begin{pmatrix} \langle g_{23}(z) \rangle \\ \langle g_{13}(z) \rangle \\ \langle g_{12}(z) \rangle \end{pmatrix} = 0,$$

so that

$$\left( d_{23} - \frac{d_2c_{23}^{(2)}}{c_2^{(2)}} - \frac{d_3c_{23}^{(3)}}{c_3^{(3)}}, d_{13} - \frac{d_1c_{13}^{(1)}}{c_1^{(1)}} - \frac{d_3c_{13}^{(3)}}{c_3^{(3)}}, d_{12} - \frac{d_1c_{12}^{(1)}}{c_1^{(1)}} - \frac{d_2c_{12}^{(2)}}{c_2^{(2)}} \right) \begin{pmatrix} \langle g_{23}(z) \rangle \\ \langle g_{13}(z) \rangle \\ \langle g_{12}(z) \rangle \end{pmatrix} = 0.$$

Therefore, putting

$$(7.40) \quad C'_{ij} = d_{ij} - c_{ij}^{(i)} \frac{d_i}{c_i^{(i)}} - c_{ij}^{(j)} \frac{d_j}{c_j^{(j)}} \quad (1 \leq i < j \leq 3),$$

we obtain

$$(7.41) \quad C'_{23} \langle g_{23}(z) \rangle + C'_{13} \langle g_{13}(z) \rangle + C'_{12} \langle g_{12}(z) \rangle = 0.$$

From direct calculation of (7.40), we have

$$(7.42) \quad C'_{23} = \frac{(1 - a_1^2)(1 - qa_1^2) \prod_{k=4}^6 (1 - a_k a_1) \prod_{4 \leq i < j \leq 6} (1 - a_1 a_i a_j)}{q^{\frac{1}{2}} a_1^2 a_4 a_5 a_6 (1 - a_1) (1 - a_1^2/a_2^2) (1 - a_1^2/a_3^2) (1 - qa_2 a_3)},$$

$$(7.43) \quad C'_{13} = \frac{(1 - a_2^2)(1 - qa_2^2) \prod_{k=4}^6 (1 - a_k a_2) \prod_{4 \leq i < j \leq 6} (1 - a_2 a_i a_j)}{q^{\frac{1}{2}} a_2^2 a_4 a_5 a_6 (1 - a_2) (1 - a_2^2/a_1^2) (1 - a_2^2/a_3^2) (1 - qa_1 a_3)},$$

$$(7.44) \quad C'_{12} = \frac{(1 - a_3^2)(1 - qa_3^2) \prod_{k=4}^6 (1 - a_k a_3) \prod_{4 \leq i < j \leq 6} (1 - a_3 a_i a_j)}{q^{\frac{1}{2}} a_3^2 a_4 a_5 a_6 (1 - a_3) (1 - a_3^2/a_1^2) (1 - a_3^2/a_2^2) (1 - qa_1 a_2)},$$

which will be confirmed just below. If we put

$$C_{ij} = C'_{ij} (q^{\frac{1}{2}} a_4 a_5 a_6) (1 - qa_1 a_2) (1 - qa_1 a_3) (1 - qa_2 a_3),$$

then, from (7.41) we can obtain our conclusion (7.21).

Lastly, we confirm the explicit forms of  $C'_{ij}$  in (7.42)–(7.44). Without loss of generality, we confirm (7.44) of  $C'_{12}$ . Since  $c_{12}^{(i)}$ ,  $c_1^{(i)}$ ,  $d_{12}$ ,  $d_1$  are given as (7.27), (7.30), (7.36), (7.39), respectively, using

$$\frac{d_1}{c_1^{(1)}} = \frac{\phi(q^{\frac{1}{2}} a_2^2/a_3, a_3/a_2)}{\varphi_1(q^{\frac{1}{2}} a_2^2/a_3, a_3/a_2)}, \quad \frac{d_2}{c_2^{(2)}} = \frac{\phi(q^{\frac{1}{2}} a_1^2/a_3, a_3/a_1)}{\varphi_2(q^{\frac{1}{2}} a_1^2/a_3, a_3/a_1)}$$

(7.40) of  $C'_{12}$  is written as

$$(7.45) \quad C'_{12} = d_{12} - c_{12}^{(1)} \frac{d_1}{c_1^{(1)}} - c_{12}^{(2)} \frac{d_2}{c_2^{(2)}}$$

$$(7.46) \quad = \frac{F_+(\mathfrak{p}_{34})}{g_{12}(\mathfrak{p}_{34})} \left[ \phi(q^{\frac{1}{2}} a_3/a_4, a_4) \right.$$

$$(7.47) \quad \left. - \varphi_1(q^{\frac{1}{2}} a_3/a_4, a_4) \frac{\phi(q^{\frac{1}{2}} a_2^2/a_3, a_3/a_2)}{\varphi_1(q^{\frac{1}{2}} a_2^2/a_3, a_3/a_2)} \right.$$

$$(7.48) \quad \left. - \varphi_2(q^{\frac{1}{2}} a_3/a_4, a_4) \frac{\phi(q^{\frac{1}{2}} a_1^2/a_3, a_3/a_1)}{\varphi_2(q^{\frac{1}{2}} a_1^2/a_3, a_3/a_1)} \right].$$

We first compute the first term of  $C'_{12}$  corresponding to (7.46) as

$$(7.49) \quad \begin{aligned} & \phi(q^{\frac{1}{2}} a_3/a_4, a_4) \\ &= (qa_3^2 a_4)^{-\frac{1}{2}} (1 + q^{\frac{1}{2}} a_3) (1 + q^{\frac{1}{2}} a_3 a_4) e(a_1, a_4) e(a_2, a_4) e(a_3, a_4) \\ &= q^{-\frac{1}{2}} a_3^{-1} (1 + q^{\frac{1}{2}} a_3) (1 + q^{\frac{1}{2}} a_3 a_4) a_4^{-\frac{1}{2}} e(a_1, a_4) e(a_2, a_4) e(a_3, a_4). \end{aligned}$$

We next compute the second term of  $C'_{12}$  corresponding to (7.47). Using

$$\begin{aligned} \frac{\phi(z)}{\varphi_i(z)} &= \frac{(z_1^2 z_2^3)^{-\frac{1}{2}} (1 + z_1 z_2) (1 + z_1 z_2^2) e(a_1, z_2) e(a_2, z_2) e(a_3, z_2)}{z_2^{-\frac{1}{2}} (1 + z_2) e(a_1, z_2) e(a_2, z_2) e(a_3, z_2) e(q^{\frac{1}{2}} a_i, z_1 z_2) e(q^{\frac{1}{2}} a_i, z_1 z_2^2)} \\ &= \frac{z_1^{-1} z_2^{-1} (1 + z_1 z_2) (1 + z_1 z_2^2)}{(1 + z_2) e(q^{\frac{1}{2}} a_i, z_1 z_2) e(q^{\frac{1}{2}} a_i, z_1 z_2^2)}, \end{aligned}$$





$$\begin{aligned}
& \times (-1)q^{-\frac{1}{2}}a_3(1-a_2/a_1)(1-a_3a_4/a_2)(1-a_1/a_3a_4)(1-q^{\frac{1}{2}}a_3)(1+qa_1a_2a_4) \\
= & \frac{F_+(\mathbb{P}_{34})}{g_{12}(\mathbb{P}_{34})}a_4^{-\frac{1}{2}}e(a_1, a_4)e(a_2, a_4)e(a_3, a_4) \\
& \times (-1)\frac{(1-a_3a_4/a_1)(1-a_3a_4/a_2)(1-qa_3^2)(1+qa_1a_2a_4)}{q^{\frac{1}{2}}a_3a_4(1+a_3/a_1)(1+a_3/a_2)(1-qa_1a_2)} \\
= & (a_3^2a_4)^{-\frac{3}{2}}\frac{\prod_{k=1}^6(1-a_k a_3)(1-a_k a_3 a_4)}{(1-a_3)(1-a_3a_4)(1-a_3/a_4)(1-a_3a_4^2)(1-a_3^2a_4)} \\
& \times \frac{a_4^{-\frac{1}{2}}e(a_1, a_4)e(a_2, a_4)e(a_3, a_4)}{e(a_1, a_3)e(a_1, a_4)e(a_1, a_3a_4)e(a_2, a_3)e(a_2, a_4)e(a_2, a_3a_4)} \\
& \times (-1)\frac{(1-a_3a_4/a_1)(1-a_3a_4/a_2)(1-qa_3^2)(1+qa_1a_2a_4)}{q^{\frac{1}{2}}a_3a_4(1+a_3/a_1)(1+a_3/a_2)(1-qa_1a_2)} \\
= & (a_3^2a_4)^{-\frac{3}{2}}\frac{(1-a_1a_3)(1-a_2a_3)(1-a_3^2)(1-a_3a_4a_5)(1-a_3a_4a_6)}{(1-a_3)a_3e(a_3, a_4)} \\
& \times \frac{a_4^{-\frac{1}{2}}e(a_1, a_4)e(a_2, a_4)e(a_3, a_4)\prod_{k=4}^6(1-a_k a_3)}{e(a_1, a_3)e(a_1, a_4)e(a_1, a_3a_4)e(a_2, a_3)e(a_2, a_4)e(a_2, a_3a_4)} \\
& \times (-1)\frac{a_3^2a_4^2e(a_3a_4, a_1)e(a_3a_4, a_2)(1-qa_3^2)(1+qa_1a_2a_4)}{q^{\frac{1}{2}}a_3a_4(1+a_3/a_1)(1+a_3/a_2)(1-qa_1a_2)} \\
= & \frac{(1-a_1a_3)(1-a_2a_3)(1-a_3^2)(1-a_3a_4a_5)(1-a_3a_4a_6)\prod_{k=4}^6(1-a_k a_3)}{(1-a_3)} \\
& \times \frac{a_3^{-3}a_4^{-1}}{e(a_1, a_3)e(a_2, a_3)} \times (-1)\frac{(1-qa_3^2)(1-a_3^{-1}a_5^{-1}a_6^{-1})}{q^{\frac{1}{2}}(1+a_3/a_1)(1+a_3/a_2)(1-qa_1a_2)} \\
& [[\text{used the balancing condition only here for } 1+qa_1a_2a_4 = 1-a_3^{-1}a_5^{-1}a_6^{-1}]] \\
= & \frac{(1-a_1a_3)(1-a_2a_3)(1-a_3^2)(1-a_3a_4a_5)(1-a_3a_4a_6)\prod_{k=4}^6(1-a_k a_3)}{(1-a_3)} \\
& \times \frac{a_3^{-4}a_4^{-1}a_5^{-1}a_6^{-1}}{e(a_1, a_3)e(a_2, a_3)} \times \frac{(1-qa_3^2)(1-a_3a_5a_6)}{q^{\frac{1}{2}}(1+a_3/a_1)(1+a_3/a_2)(1-qa_1a_2)} \\
= & (1-a_1a_3)(1-a_2a_3)\prod_{k=4}^6(1-a_k a_3)\prod_{4\leq i<j\leq 6}(1-a_3a_i a_j) \\
& \times \frac{a_3^{-4}a_4^{-1}a_5^{-1}a_6^{-1}}{a_3^{-2}(1-a_1a_3)(1-a_3/a_1)(1-a_2a_3)(1-a_3/a_2)} \\
& \times \frac{(1-a_3^2)(1-qa_3^2)}{q^{\frac{1}{2}}(1-a_3)(1+a_3/a_1)(1+a_3/a_2)(1-qa_1a_2)} \\
= & \frac{(1-a_3^2)(1-qa_3^2)\prod_{k=4}^6(1-a_k a_3)\prod_{4\leq i<j\leq 6}(1-a_3a_i a_j)}{q^{\frac{1}{2}}a_3^2a_4a_5a_6(1-a_3)(1-a_3^2/a_1^2)(1-a_3^2/a_2^2)(1-qa_1a_2)},
\end{aligned}$$

which coincide with (7.44). □

## APPENDIX A. FOUR-TERM IDENTITY

In this appendix we show an identity of four terms, which is used in (7.52).

**Lemma A.1.** *The following identity holds for  $x_1, x_2, x_3, a$ :*

$$\begin{aligned} & x_1(1-x_1)(a-x_2)(a-x_3)(1-x_2x_3)(x_3-x_2) \\ & \quad + x_2(1-x_2)(a-x_1)(a-x_3)(1-x_1x_3)(x_1-x_3) \\ & \quad + x_3(1-x_3)(a-x_1)(a-x_2)(1-x_1x_2)(x_2-x_1) \\ & = (x_1-x_2)(x_2-x_3)(x_3-x_1)(1-a)(a-x_1x_2x_3), \end{aligned}$$

which is equivalent to

$$\begin{aligned} & a(1-x_1)(1-x_2/a)(1-x_3/a)e(x_2, x_3) \\ & \quad + a(1-x_2)(1-x_1/a)(1-x_3/a)e(x_3, x_1) \\ (A.1) \quad & \quad + a(1-x_3)(1-x_1/a)(1-x_2/a)e(x_1, x_2) \\ & = (1-x_1/x_2)(1-x_2/x_3)(1-x_3/x_1)(1-a)(1-x_1x_2x_3/a). \end{aligned}$$

**Proof.** The proof follows by direct computation. □

**Remark.** In the computation (7.52) we used (A.1) with

$$x_1 = -q^{\frac{1}{2}}a_1, \quad x_2 = -q^{\frac{1}{2}}a_2, \quad x_3 = -q^{\frac{1}{2}}a_3a_4, \quad a = q^{\frac{1}{2}}a_3.$$

**Remark.** The above identity is a special case ( $q \rightarrow 0$ ) of the following four-term identity satisfied by  $\vartheta(x; q)$ :

$$\begin{aligned} & ax_1x_3\vartheta(-q^{\frac{1}{2}}x_1; q)\vartheta(-q^{\frac{1}{2}}x_1/a; q)\vartheta(x_1; q)\vartheta(x_2/a; q)\vartheta(x_3/a; q)\vartheta(x_2x_3; q)\vartheta(x_2/x_3; q) \\ & \quad + ax_1x_2\vartheta(-q^{\frac{1}{2}}x_2; q)\vartheta(-q^{\frac{1}{2}}x_2/a; q)\vartheta(x_2; q)\vartheta(x_1/a; q)\vartheta(x_3/a; q)\vartheta(x_1x_3; q)\vartheta(x_3/x_1; q) \\ & \quad + ax_2x_3\vartheta(-q^{\frac{1}{2}}x_3; q)\vartheta(-q^{\frac{1}{2}}x_3/a; q)\vartheta(x_3; q)\vartheta(x_1/a; q)\vartheta(x_2/a; q)\vartheta(x_1x_2; q)\vartheta(x_1/x_2; q) \\ & = x_1x_2x_3\theta(-q^{\frac{1}{2}}; q)\theta(-q^{\frac{1}{2}}a; q)\vartheta(x_1/x_2; q)\vartheta(x_2/x_3; q)\vartheta(x_3/x_1; q)\vartheta(a; q)\vartheta(x_1x_2x_3/a; q) \end{aligned}$$

We omit the details.

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## REFERENCES

1. R. Askey: Some basic hypergeometric extensions of integrals of Selberg and Andrews. *SIAM J. Math. Anal.* **11** (1980), 938–951.
2. R. Askey and J. Wilson: Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials. *Mem. Amer. Math. Soc.* **54** (1985), no. 319, iv+55 pp.
3. P. J. Forrester and S. O. Warnaar: The importance of the Selberg integral, *Bull. Amer. Math. Soc. (N.S.)* **45** (2008), 489–534.
4. G. Gaspar and M. Rahman: *Basic hypergeometric series*, 2nd ed., Cambridge University Press, Cambridge, 2004.
5. R. A. Gustafson: Some  $q$ -beta and Mellin–Barnes integrals on compact Lie groups and Lie algebras. *Trans. Amer. Math. Soc.* **341** (1994), 69–119.
6. R. A. Gustafson: A summation theorem for hypergeometric series very-well-poised on  $G_2$ , *SIAM J. Math. Anal.* **21** (1990), 510–522.

7. M. Ito: Three-term relations between interpolation polynomials for a  $BC_n$ -type basic hypergeometric series, *Adv. Math.* **226** (2011), 4096–4130.
8. M. Ito: Askey–Wilson type integrals associated with root systems. *Ramanujan J.* **12** (2006), 131–151.
9. M. Ito and P. J. Forrester: A bilateral extension of the  $q$ -Selberg integral, *Trans. Amer. Math. Soc.* **369** (2017), 2843–2878.
10. M. Ito and M. Noumi: Elliptic extension of Gustafson’s  $q$ -integral of type  $G_2$ , *Adv. Math.* **370** (2020), 107211, 40 pp.
11. M. Ito and M. Noumi: Evaluation of the  $BC_n$  elliptic Selberg integral via the fundamental invariants. *Proc. Amer. Math. Soc.* **145** (2017), 689–703.
12. I. G. Macdonald: *Affine Hecke algebras and orthogonal polynomials*. Cambridge Tracts in Mathematics, 157. Cambridge University Press, Cambridge, 2003. x+175 pp.
13. B. Nassrallah and M. Rahman: Projection formulas, a reproducing kernel and a generating function for  $q$ -Wilson polynomials, *SIAM J. Math. Anal.* **16** (1985), 186–197.
14. E. M. Rains: Transformations of elliptic hypergeometric integrals. *Ann. of Math. (2)* **171** (2010), 169–243.
15. V. P. Spiridonov: Short proofs of the elliptic beta integrals. *Ramanujan J.* **13** (2007), 265–283.
16. Y. Takushi: An infinite product expression for the  $q$ -hypergeometric integral of type  $G_2$  and its extension (in Japanese), Master’s Thesis, Department of Mathematical Sciences, University of the Ryukyus, March 2020.
17. J. F. van Diejen and V. P. Spiridonov: Elliptic Selberg integrals. *Internat. Math. Res. Notices* 2001. 1083–1110.

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