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Inversions and Möbius invariants

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Abstract

Two n-point-sets in Euclidean space are said to be inversion-equivalent if one set can be transformed into the other set by applying inversions of the space. All 3-point-sets are inversion-equivalent to each other. For each four points x, y, z, w in an n-point-set, $n \geq 4$, the ratio $(xy \cdot zw)/(xw \cdot yz)$ is invariant under inversions, which is called a Möbius invariant of the n-point-set. We prove that for $4 \leq n \leq d+2$, the minimum number of Möbius invariants necessary to determine all Möbius invariants for every n-point-set in Euclidean d-space is equal to $n(n-3)/2$, and discuss the case of planar n-point-sets in some detail. We also characterize those fractional functions that are invariant under inversions.

1 Introduction

Let S be a sphere in the d-dimensional Euclidean space $\mathbb{R}^d$ with center p and radius r. The *inversion* of $\mathbb{R}^d$ with respect to S is the transformation of $\mathbb{R}^d$ that sends each point $x\ (\neq p)$ to a point x' on the ray $\overrightarrow{px}$ such that $px \cdot px' = r^2$, where px denotes the distance between p and x. The point p and the radius r are called the *center* and the *radius* of the inversion, respectively. Note that in an inversion, the image of its center is not defined. One of the typical features of an inversion is that it transforms a sphere into another sphere, with regarding a hyperplane as a sphere of infinite radius. For more about inversions, see, e.g. [2,5,8].

In this paper, we consider to transform a finite point-set by inversions. Note that *an inversion can be applied to a point-set only when*

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its center does not belong to the point-set, since the image of the center of an inversion is not defined. (Usually, to avoid such restriction, a point at infinity is added to $\mathbb{R}^d$ as to be the image of the center for every inversion. Another less usual way is to define the image of the center of an inversion to be the center itself. In [7], transformations of a finite point-set by 'center-fixing-inversions' with centers in the point-set are investigated.)

A pair of n-point-sets are called *inversion-equivalent* if one set can be transformed into the other set by applying a series of inversions. This relation is clearly an equivalence relation. An *ordered n-point-set* is an n-point-set whose points are ordered, which is denoted by the juxtaposition of the n points in order like $\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3\ldots\boldsymbol{a}_n$ (the same notation as used for a polygon or a polygonal curve). Two ordered n-point-sets are called *inversion-equivalent* if they are inversion-equivalent with keeping the order. It turns out that all ordered triples are mutually inversion-equivalent.

For a quadruplet (that is, an ordered 4-point-set) $\boldsymbol{abcd}$, let us define $[\boldsymbol{abcd}]$ by

$$[\boldsymbol{abcd}] = (\boldsymbol{ab}\cdot\boldsymbol{cd})/(\boldsymbol{ad}\cdot\boldsymbol{bc}),$$

which is called a *Möbius invariant* of the quadruplet. This is indeed invariant under any inversion (Corollary 2.1, see also [5, p.92], [3, p.310]). Hence, for example, a quadruplet $\boldsymbol{abcd}$ with $[\boldsymbol{abcd}] \neq 1$ is never inversion-equivalent to the vertex set of a regular tetrahedron.

For a quadruplet $\boldsymbol{a}_i\boldsymbol{a}_j\boldsymbol{a}_k\boldsymbol{a}_l$ taken from an ordered n-point-set $\alpha = \boldsymbol{a}_1\boldsymbol{a}_2\ldots\boldsymbol{a}_n$, $n \geq 4$, the Möbius invariant $[\boldsymbol{a}_i\boldsymbol{a}_j\boldsymbol{a}_k\boldsymbol{a}_l]$ is simply denoted by $[ijkl]$, and its 'value' at α is denoted by $[ijkl]_\alpha$. Let us call

$$[ijkl] \quad (i,j,k,l \text{ are all different, } 1 \leq i,j,k,l \leq n)$$

the Möbius invariants for an ordered n-point-set. We show the following:

- *For $n \geq 4$, two ordered n-point-sets α, β are inversion-equivalent if and only if $[ijkl]_\alpha = [ijkl]_\beta$ holds for every $[ijkl]$.*

Applying this result, it is proved that every quadruplet is inversion equivalent to the vertex set of a (possibly degenerate) parallelogram.

Notice that $[ijkl] = [klij] = [jilk] = [lkji]$. Hence there are at most $6\binom{n}{4}$ distinct Möbius invariants for an ordered n-point-set.

They are not independent as 'variables'. Rather a few of them determine all others. Let $R(d,n)$ denote the minimum cardinality of a set of Möbius invariants whose values determine the values of all Möbius invariants for every ordered n-point-set in $\mathbb{R}^d$. We prove the following.

- $R(d,n) = n(n-3)/2$ for $4 \leq n \leq d+2$.

If $d \ll n$ then $R(d,n)$ would be much smaller than $n(n-3)/2$ by the dimensional restriction. However, it is not easy to determine the value of $R(d,n)$ even in the planar case $d = 2$, where the bound I could prove is $R(2,n) \leq 3n - 10$. In Section 5, we will discuss the planar case in some detail.

Besides Möbius invariants, there are many fractional functions, such as $(\boldsymbol{ab} \cdot \boldsymbol{ac} \cdot \boldsymbol{de})/(\boldsymbol{ad} \cdot \boldsymbol{ae} \cdot \boldsymbol{bc})$ and $(\boldsymbol{ab} \cdot \boldsymbol{cd} \cdot \boldsymbol{ef})/(\boldsymbol{bc} \cdot \boldsymbol{de} \cdot \boldsymbol{fa})$ that are invariant under inversions. They are characterized in the following way:

- *A fractional function is invariant under inversions if and only if, for each point-symbol, the number of times it appears in the numerator is equal to the number of times it appears in the denominator.*

2 A few basic facts on inversions

Lemma 2.1. *Let $\boldsymbol{a}', \boldsymbol{b}'$ denote the images of $\boldsymbol{a}, \boldsymbol{b}$ under the inversion with center $\boldsymbol{p}$ and radius r. Then $\boldsymbol{a}'\boldsymbol{b}' = (r^2 \cdot \boldsymbol{ab})/(\boldsymbol{pa} \cdot \boldsymbol{pb})$.*

Proof. Since $\boldsymbol{pa}' \cdot \boldsymbol{pa} = r^2 = \boldsymbol{pb}' \cdot \boldsymbol{pb}$ and $\angle \boldsymbol{apb} = \angle \boldsymbol{a}'\boldsymbol{pb}'$, the two (possibly degenerate) triangles $\boldsymbol{pa}'\boldsymbol{b}'$ and $\boldsymbol{pba}$ are similar. Hence $\boldsymbol{a}'\boldsymbol{b}'/\boldsymbol{ba} = \boldsymbol{pa}'/\boldsymbol{pb} = r^2/(\boldsymbol{pa} \cdot \boldsymbol{pb})$, and the lemma follows. □

The following corollaries follow from this by simple calculations.

Corollary 2.1. *Suppose an inversion sends $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ to $\boldsymbol{a}', \boldsymbol{b}', \boldsymbol{c}', \boldsymbol{d}'$, respectively. Then $[\boldsymbol{a}'\boldsymbol{b}'\boldsymbol{c}'\boldsymbol{d}'] = [\boldsymbol{abcd}]$.* □

Corollary 2.2. *Let f_1, f_2 be the inversions with the same center $\boldsymbol{p}$ and radii r_1, r_2, respectively. Then the composition $f_2 \circ f_1$ is a homothety with center $\boldsymbol{p}$ and similitude ratio $(r_2/r_1)^2$.* □

Lemma 2.2. *Let H be a hyperplane in $\mathbb{R}^d$, $p \in \mathbb{R}^d \setminus H$, and $q \in \mathbb{R}^d$ be the point that is symmetric to p with respect to H. Let g_1 be the inversion of $\mathbb{R}^d$ with respect to the sphere with center p and radius pq, and let g_2 be the inversion of $\mathbb{R}^d$ with center q and radius pq. Then the composition $f := g_1 \circ g_2 \circ g_1$ is the reflection of $\mathbb{R}^d$ with respect to the hyperplane H.*

Proof. It will be enough to consider the plane case $d = 2$. We may suppose that $p = (-1,0), q = (1,0)$ and H is the y-axis. Note that $g_1(q) = q$, $g_2(p) = p$. Let C_q be the circle with center q and radius pq. For a point u on the y-axis, let v be the intersection of the ray $\overrightarrow{pu}$ and C_q other than p. Then, since g_1 sends the y-axis to C_q, we have $g_1(u) = v$. Since $g_2(v) = v$, we have $f(u) = u$. Thus, f fixes all points on the y-axis. It is also clear that f sends the x-axis to itself.

For any line ℓ, $g_1(\ell)$ is either a line passing through p or a circle passing through p, and hence, $g_2(g_1(\ell))$ is either a circle passing through p or a line passing through p. Therefore, $g_1(g_2(g_1(\ell)))$ is always a line. Thus, f sends every line to a line, and sends every pair of parallel lines to a pair of parallel lines.

Now, since $g(x,0) = (\frac{3-x}{x+1}, 0)$, $g_2(x,0) = (\frac{x+3}{x-1}, 0)$ as easily verified, $f(x,0) = (-x,0)$ follows by a simple calculation. Then, for a given point (x_0, y_0), the line "$x = x_0$" (which is parallel to the y-axis) is sent to the line passing through $(-x_0, 0)$ and parallel to the y-axis, that is, the line "$x = -x_0$". Similarly, the line "$y = y_0$" is sent to itself by f. Therefore the intersection (x_0, y_0) of the two lines $x = x_0$ and $y = y_0$ is sent to $(-x_0, y_0)$, that is, $f(x_0, y_0) = (-x_0, y_0)$. This proves the lemma. □

Note that in Corollary 2.2, the center of a homothety can be chosen independently from the similitude ratio, and in Lemma 2.2, the point $p \in \mathbb{R}^d \setminus H$ can be chosen arbitrarily. Since all reflections of $\mathbb{R}^d$ generate all isometries of $\mathbb{R}^d$, we have the following corollary from Corollary 2.2 and Lemma 2.2.

Corollary 2.3. *If two n-point-sets are similar, then they are inversion-equivalent.* □

Lemma 2.3. *Let σ be a finite point-set containing three points a, b, c. Then, for every $\lambda > \lambda_0$ (where λ_0 is a constant depending on σ), there exists an inversion with center $p \notin \sigma$ that transforms a, b, c into a', b', c' such that $a'b' = \lambda$, $b'c' = 1$, $a'c' = 1 + \lambda$.*

Proof. Let Γ be the circle (or line) passing through a, b, c. Let p be a point on $\Gamma \backslash$(the arc $\widehat{abc}$). Let f be an inversion with center p and some radius r, and let $a' = f(a), b' = f(b), c' = f(c)$. Then, a', b', c' are collinear in this order, and by Lemma 2.1, we have

$$\frac{a'b'}{b'c'} = \left(\frac{r^2 \cdot ab}{pa \cdot pb}\right)\left(\frac{pb \cdot pc}{r^2 \cdot bc}\right) = \frac{ab \cdot pc}{bc \cdot pa}.$$

Let ε be the distance from a to the nearest point in $\sigma - \{a\}$. Then, $\varepsilon > 0$. Let $\lambda_0 = (ab/bc)(ac/\varepsilon + 1)$. If $pa = \varepsilon$, we have

$$\frac{ab \cdot pc}{bc \cdot pa} \leq \frac{ab(pa + ac)}{bc \cdot pa} = \frac{ab}{bc}\left(\frac{ac}{\varepsilon} + 1\right) = \lambda_0.$$

Since pc/pa continuously tends to infinity as pa continuously tends to 0, $(ab \cdot pc)/(bc \cdot pa)$ can take every value $\lambda \geq \lambda_0$. Thus, for every $\lambda > \lambda_0$, we can choose p on Γ so that $pa < \varepsilon$, $a'b'/b'c' = (ab/bc)(pc/pa) = \lambda$, and we can choose $r > 0$ so that $b'c' = 1$. □

An ordered triple abc is called a *linear triple* if a, b, c are collinear in this order.

Corollary 2.4. *Every pair of ordered triples are inversion-equivalent.* □

3 Möbius invariants

For three points $a, b, p \in \mathbb{R}^d$ $(d \geq 2)$, the locus of the points x satisfying $ax/bx = ap/bp$ is called the *Apollonian sphere* (*Apollonian circle* if $d = 2$) determined by ab and p, which is denoted by $A(ab, p)$. If $ap \neq bp$, then $A(ab, p)$ is indeed a sphere with center at the extension of the line segment ab (beyond a or b), but if $ap = bp$, then $A(ab, p)$ is a hyperplane that bisects the line segment ab perpendicularly. Since

$$A(ab, p) \ni q \Leftrightarrow \frac{ap}{bp} = \frac{aq}{bq} \Leftrightarrow \frac{pa}{qa} = \frac{pb}{qb} \Leftrightarrow A(pq, a) \ni b$$

holds, $q \in A(ab, p) \cap A(ac, p)$ implies that $\{a, b, c\} \subset A(pq, a)$. Therefore, *if* abc *is a linear triple, then* $A(ab, p)$ *and* $A(ac, p)$ *are different spheres.* For more about Apollonian circles, see Coxeter [5].

For an ordered n-point-set α, we denote the distance between the i-th point and the j-th point by d_{ij} or $d_{ij}(\alpha)$.

Lemma 3.1. *Let $n \geq 4$ and $\alpha = a_1 a_2 \ldots a_n$ be an ordered n-point-set in which $a_1 a_2 a_3$ is a linear triple. Then the two distances d_{12}, d_{23} and the values of the Möbius invariants*

$$[j123], [j213], \ j = 4, 5, \ldots, n, \text{ and } [jk12], \ 4 \leq j < k \leq n, \tag{1}$$

determine all the distances d_{ij} in α.

Proof. Suppose that three points a_1, a_2, a_3 are fixed so that $d_{12} = s, d_{23} = t, d_{13} = s + t$. Since $[4213]_\alpha = (a_4 a_2 \cdot (s+t))/(a_4 a_3 \cdot s)$, we have $a_2 a_4 / a_3 a_4 = [4213]_\alpha \cdot s/(s+t)$. Hence the value of [4213] determines the Apollonian sphere $A(a_2 a_3, a_4)$. Similarly, the value of [4123] determines the Apollonian sphere $A(a_1 a_3, a_4)$. Since $a_1 a_2 a_3$ is a linear triple, these two Apollonian spheres are different with centers on the line $a_1 a_2$. Hence, all intersection points of the two Apollonian spheres are at the same distance from the line $a_1 a_2$, and hence each of the distances d_{41}, d_{42}, d_{43} are uniquely determined. Similarly, for each $4 \leq j \leq n$, the values of $[j213]$ and $[j123]$ determine the distances d_{j1}, d_{j2}, d_{j3} uniquely. Then, the values of $[jk12]$ determine the distances d_{jk} for $4 \leq j < k \leq n$. □

Let $\alpha = a_1 \ldots a_n$ be an ordered n-point-set in the plane $\mathbb{R}^2$, $n \geq 4$. Suppose $a_1 = (-s, 0), a_2 = (0,0), a_3 = (t, 0)$ and $a_4 = (u, v)$, $uv \neq 0$. Let f be the inversion with center $(0, w)$, radius r, and let g be the inversion with the center $(0, -w)$ and radius r, where $w \neq 0$. Then the triples $f(a_1 a_2 a_3)$, $g(a_1 a_2 a_3)$ are congruent, and every Möbius invariant takes the same value at $f(\alpha)$ and at $g(\alpha)$. But $f(\alpha)$ and $g(\alpha)$ are not congruent. Let us state this fact as a remark.

Remark 3.1. If the first three points of an ordered n-point-set ($n \geq 4$) in $\mathbb{R}^2$ are not collinear, then the three distances d_{12}, d_{23}, d_{31} and the values of all Möbius invariants are not enough to determine all distances among the n points.

Theorem 3.1. *For every $n \geq 4$, a pair of ordered n-point-sets α and β in $\mathbb{R}^d$ are inversion-equivalent if and only if $[ijkl]_\alpha = [ijkl]_\beta$ holds for every $[ijkl]$.*

Proof. The *if* part is obvious since Möbius invariants are invariant under inversions.

Let $\alpha = a_1 \ldots a_n$, $\beta = b_1 \ldots b_n$. By Lemma 2.3, we can apply inversions to α and β independently, so that $a_1a_2a_3$ and $b_1b_2b_3$ become congruent linear triples. Hence, we may assume from the first that $a_1a_2a_3$ and $b_1b_2b_3$ are congruent linear triples, $d_{12}(\alpha) = d_{12}(\beta) = s$, $d_{23}(\alpha) = d_{23}(\beta) = t$. Since $[ijkl]_\alpha = [ijkl]_\beta$ for every $[ijkl]$, it follows that $d_{ij}(\alpha) = d_{ij}(\beta)$ holds for all i, j by Lemma 3.1. Hence the two ordered n-point-sets are congruent to each other. Therefore, they are inversion-equivalent by Corollary 2.3. □

Theorem 3.2. *Every quadruplet in $\mathbb{R}^2$ is inversion-equivalent to the vertex set of a (possibly degenerate) parallelogram.*

Proof. Let $\boldsymbol{abcd}$ be a quadruplet, and put $a = [\boldsymbol{abcd}], b = [\boldsymbol{acbd}]$. (Then, $[\boldsymbol{abdc}] = a/b$, $[\boldsymbol{acdb}] = b/a$, $[\boldsymbol{adbc}] = 1/b$, $[\boldsymbol{adcb}] = 1/a$ as easily verified.) By generalized Ptolemy's inequality (see, e.g. [1]) we have

$$\begin{aligned} \boldsymbol{ab}\cdot\boldsymbol{cd} &\leq \boldsymbol{ac}\cdot\boldsymbol{bd} + \boldsymbol{bc}\cdot\boldsymbol{ad}, \\ \boldsymbol{ac}\cdot\boldsymbol{bd} &\leq \boldsymbol{ab}\cdot\boldsymbol{cd} + \boldsymbol{ad}\cdot\boldsymbol{bc}, \\ \boldsymbol{ad}\cdot\boldsymbol{bc} &\leq \boldsymbol{ab}\cdot\boldsymbol{cd} + \boldsymbol{ac}\cdot\boldsymbol{bd}. \end{aligned}$$

Since $[\boldsymbol{abcd}] = (\boldsymbol{ab}\cdot\boldsymbol{cd})/(\boldsymbol{bc}\cdot\boldsymbol{ad})$ and $[\boldsymbol{acbd}] = (\boldsymbol{ad}\cdot\boldsymbol{bd})/(\boldsymbol{ad}\cdot\boldsymbol{bc})$, it follows that $a \leq b+1, b \leq a+1$ and $1 \leq a+b$. Therefore $|a-1| \leq b \leq a+1$. Let

$$p = \pm\frac{1}{2}\sqrt{(a+1)^2 - b^2},\ q = \frac{1}{2}\sqrt{b^2 - (a-1)^2}$$

(we may choose either sign $\pm$ for p) and put

$$\boldsymbol{a}' = (0,0),\ \boldsymbol{b}' = (p,q),\ \boldsymbol{c}' = (p+1,q),\ \boldsymbol{d}' = (1,0) \in \mathbb{R}^2.$$

Then $\boldsymbol{a}'\boldsymbol{b}'\boldsymbol{c}'\boldsymbol{d}'$ is a parallelogram, and

$$[\boldsymbol{a}'\boldsymbol{b}'\boldsymbol{c}'\boldsymbol{d}'] = a,\ [\boldsymbol{a}'\boldsymbol{c}'\boldsymbol{b}'\boldsymbol{d}'] = b.$$

Hence $\boldsymbol{abcd}$ and $\boldsymbol{a}'\boldsymbol{b}'\boldsymbol{c}'\boldsymbol{d}'$ are inversion-equivalent. □

Corollary 3.1. *The vertex-sets of two parallelograms are not inversion-equivalent unless the two parallelograms are similar to each other.* □

Since a parallelogram $\boldsymbol{abcd}$ is a rhombus if and only if $[\boldsymbol{abcd}] = 1$, we have the following.

Corollary 3.2. *A quadruplet $\boldsymbol{abcd}$ is inversion-equivalent to the vertex set of a rhombus if and only if $[\boldsymbol{abcd}] = 1$.* □

4 Number of necessary invariants

For a quadruplet $a_1a_2a_3a_4$ in $\mathbb{R}^d$ $(d > 0)$, let $x = d_{12}$, $y = d_{23}$, $z = d_{34}$, $w = d_{41}$. Then $[1234] = xz/(yw)$, which is the ratio of the two products of opposite edges in the (possibly self-intersecting) quadrilateral. By changing the order of the vertices cyclically, we get two distinct Möbius invariants, namely, $[1234] = [3412] = xz/(yw)$ and $[2341] = [4123] = yw/(xz) = [1234]^{-1}$. Since four points produce three distinct quadrilaterals, it follows that there are $6 \times \binom{n}{4}$ distinct Möbius invariants for n points. Since $[2341] = [1234]^{-1}$, half of the $6\binom{n}{4}$ Möbius invariants are reciprocals of the other half. So, $6\binom{n}{4}$ Möbius invariants are determined by $3\binom{n}{4}$ members, probably much fewer members. Recall that $R(d,n)$ denotes the minimum cardinality of a set of Möbius invariants whose values determine the values of all Möbius invariants for every ordered n-point-set in $\mathbb{R}^d$.

Lemma 4.1. *For $d > n-2 \geq 2$, $R(d,n) = R(n-2,n)$.*

Proof. Since every n points in $\mathbb{R}^d$ lie on an $(n-1)$-dimensional flat, $R(d,n) = R(n-1,n)$ holds. Every n points in $\mathbb{R}^{n-1}$ lie on a sphere or on a hyperplane in $\mathbb{R}^{n-1}$, and every sphere can be transformed into a hyperplane (that is, an $(n-2)$-dimensional flat) by an inversion of $\mathbb{R}^{n-1}$. Hence $R(n-1,n) = R(n-2,n)$. □

Theorem 4.1. *For $4 \leq n \leq d+2$, $R(d,n) = n(n-3)/2$.*

Proof. Every ordered n-point-set is inversion-equivalent to an ordered n-point-set in which the first three points are collinear with fixed distances $d_{12} = \lambda, d_{23} = 1, d_{13} = \lambda + 1$. Then, as in Lemma 3.1, the Möbius invariants in (1) determine all the distances between the n points, and hence determine all Möbius invariants. The number of Möbius invariants in (1) is equal to $2(n-3) + \binom{n-3}{2} = n(n-3)/2$. Hence, $R(d,n) \leq n(n-3)/2$.

Next, we show that $R(d,n) \geq n(n-3)/2$. By Lemma 4.1, it is enough to show $R(n-1,n) \geq n(n-3)/2$. Let $\alpha = a_1a_2 \dots a_n$ be an ordered n-point-set in $\mathbb{R}^{n-1}$ that span an $(n-1)$-dimensional simplex. Then, every small perturbations of the distances d_{ij} in α also determine a simplex in $\mathbb{R}^{n-1}$. Hence there is a neighborhood U of the point $(\dots, d_{ij}(\alpha), \dots)$ in $\mathbb{R}^{\binom{n}{2}}$ such that every $(\dots, d_{ij}, \dots) \in U$

can be attained by an ordered n-point-set in $\mathbb{R}^{n-1}$. Let $x_{ij} = \log d_{ij}$, and $a_{ijkl}(\alpha) = \log [ijkl]_\alpha$. Then

$$\log[ijkl] = \log\left(\frac{d_{ij}d_{kl}}{d_{il}d_{jk}}\right) = x_{ij} + x_{kl} - x_{il} - x_{jk}.$$

Since the value of each d_{ij} can be changed by moving a_i continuously with keeping the values of other distances fixed, the $\binom{n}{2}$ variables d_{ij} are independent in the sense that the value of each d_{ij} is not determined by the values of all other variables. Hence the $\binom{n}{2}$ variables $x_{ij} = \log d_{ij}$ are also independent. Since $a_{ijkl}(\alpha) = \log[ijkl]_\alpha$, if we regard x_{ij}s as unknowns, the simultaneous linear equations

$$x_{ij} + x_{kl} - x_{il} - x_{jk} = a_{ijkl}(\alpha),\ 1 \leq i,j,k,l \leq n \tag{2}$$

$(i,j,k,l$ are all different) has a solution. Therefore the coefficient matrix and the 'enlarged' coefficient matrix of (2) have the same rank, say, r. Let us show that $R(n-1,n) \geq r$.

To see this, suppose that $R(n-1,n) = m < r$. We may suppose that the first m equations in the linear system (2) correspond to the m Möbius invariants. Then the coefficient matrix of the first m equations of (2) must have full rank m (for otherwise, in the first m equations of (2), some equations are obtained from others, which implies that a smaller number of Möbius invariants determine all other Möbius invariants, contradicting $R(n-1,n) = m$). Hence, by adding to these m equations $r-m$ other equations chosen from the remaining equations in (2), we can make a new system of r linear equations that has rank r. Then the new system of r linear equations determine an *onto* linear map $f : \mathbb{R}^{\binom{n}{2}} \to \mathbb{R}^r$ by $(\ldots, x_{ij}, \ldots) \mapsto (\ldots, a_{ijkl}, \ldots)$. Since every onto linear map is an *open map*, f sends every neighborhood of $(\ldots, x_{ij}(\alpha), \ldots) \in \mathbb{R}^{\binom{n}{2}}$, to a neighborhood of $(\ldots, a_{ijkl}(\alpha), \ldots) \in \mathbb{R}^r$. Hence, there is a neighborhood V of $(\ldots, a_{ijkl}(\alpha), \ldots) \in \mathbb{R}^r$ such that every $(\ldots, a_{ijkl}, \ldots)$ in V can be attained by an n-point-set in $\mathbb{R}^{n-1}$. Thus, there is an $\varepsilon > 0$ such that, in the system of r linear equations, if we replace the constant $a_{ijkl}(\alpha)$ of the last equation (that is, the lastly added equation) with $a_{ijkl}(\alpha) + \varepsilon$, then the system of the r linear equations still have a solution that can be attained by an ordered n-point-set in $\mathbb{R}^{n-1}$. This implies that the m Möbius invariants cannot determine the Möbius invariant corresponding to the last equation, a contradiction. Hence, we have $R(n-1,n) \geq r$.

Now, the coefficient vectors of the $n(n-3)/2$ equations

$$\begin{aligned} x_{j1} + x_{23} - x_{j3} - x_{12} &= \alpha_{j123},\ j = 4,5,\dots,n \\ x_{j2} + x_{13} - x_{j3} - x_{12} &= \alpha_{j213},\ j = 4,5,\dots,n \\ x_{jk} + x_{12} - x_{j2} - x_{k1} &= \alpha_{jk12},\ \leq j < k \leq n \end{aligned}$$

are linearly independent. This is shown in the following way:

Let $\vec{v}(j123), \vec{v}(j213), \vec{v}(jk12)$ be the corresponding coefficient vectors, and suppose that

$$\sum_{j=4}^{n} s_j \vec{v}(j123) + \sum_{j=4}^{n} t_j \vec{v}(j213) + \sum_{4 \leq j < k \leq n} u_{jk} \vec{v}(jk12) = 0.$$

Since each variable x_{jk} $(4 \leq j < k \leq n)$ appears in just one of the last $\binom{n-3}{2}$ equations, we must have $u_{jk} = 0$. Then, since x_{j1} and x_{j2} are independent variables, we have similarly $s_j = t_j = 0$. Therefore the rank of the coefficient vectors of (2) is at least $n(n-3)/2$. Thus $R(n-1,n) \geq n(n-3)/2$. □

5 Planar case

Lemma 5.1. *A quadruplet* $a_1 a_2 a_3 a_3$ *lie on a circle (or a line) if and only if it satisfies* $[4123] - [4213] = \pm 1$.

Proof. By Ptolemy's theorem, $a_1 a_2 a_3 a_3$ lie on a circle (or a line) if and only if $d_{41}d_{23} = d_{13}d_{42} + d_{43}d_{12}$ or $d_{42}d_{13} = d_{43}d_{12} + d_{41}d_{23}$. These equalities are equivalent to $[4123] = [4213] + 1$ or $[4213] = 1 + [4123]$, and hence, equivalent to $[4123] - [4213] = \pm 1$. □

Theorem 5.1. *For* $n \geq 4$, $R(2,n) \leq 3n - 10$.

Proof. Let $\alpha_n = a_1 a_2 \dots a_n$ denote an ordered n-point-set in the plane. Applying inversions, we may suppose that $a_1 a_2 a_3$ is a linear triple with $d_{12} = \lambda, d_{23} = 1$ for a fixed λ. If $n = 4$, then $F_2 = \{[4123], [4213]\}$ determines α_4 up to congruence, and the theorem holds. If $n = 5$, then $F_5 = \{[4123], [4213], [5123], [5213], [5412]\}$ determines α_5 up to congruence, and hence determines all Möbius invariants. Since $5 = 3 \cdot 5 - 10$, the theorem holds.

Suppose that there is a set F_{n-1} consisting of at most $3(n-1) - 10$ Möbius invariants such that (i) F_{n-1} determines α_{n-1} up to congruence, and (ii) F_{n-1} contains $[j123], [j213], 4 \leq j \leq n-1$. Then for each $4 \leq j \leq n-1$, we can check, by (ii) and Lemma 5.1, whether a_j lies on the line a_1a_2 or not. If there is an a_j $(j \leq n-1)$ that does not lie on the line a_1a_2, then put $F_n = F_{n-1} \cup \{[n123], [n213], [nj12]\}$, otherwise, put $F_n = F_{n-1} \cup \{[n123], [n213]\}$. Then, F_n determines α_n up to congruence, and F_n contains $[j123], [j213], 4 \leq j \leq n$. Since F_n contains at most $3n - 10$ members, the proof is done. □

If the first three points in an ordered n-point-set α_n $(n \geq 4)$ in $\mathbb{R}^2$ are not collinear, then the three distances d_{12}, d_{23}, d_{31} and the values of all Möbius invariants are not enough to determine all distances in α_n as pointed out in Remark 3.1.

Lemma 5.2. *If* $[4123] - [4213] \neq \pm 1$ *holds in an ordered n-point-set* $(n \geq 4)$ *in* $\mathbb{R}^2$*, then the four distances* $d_{12}, d_{23}, d_{31}, d_{14}$ *and the values of all Möbius invariants determine all the distances* d_{ij} *in the n-point-set uniquely.*

Proof. The four distances $d_{12}, d_{23}, d_{31}, d_{14}$ and $[4123], [4213]$ determine all distances among the first four points. Hence the lemma is true for $n = 4$. To show the lemma for $n > 4$, let α, β be two n-point-sets in $\mathbb{R}^2$ that have the same first four points $a_1a_2a_3a_4$. Then it will be enough to show that if $[4123] - [4213] \neq \pm 1$ holds in $a_1a_2a_3a_4$ and $[ijkl]_\alpha = [ijkl]_\beta$ for every $[ijkl]$, then $\alpha = \beta$. Let Γ be a circle passing through a_1, a_2, a_3. Let f be an inversion with center $p \in \Gamma \setminus (\alpha \cup \beta)$ and some radius r. Then, since the first three points in $f(\alpha) \cap f(\beta)$ are collinear, $d_{ij}(f(\alpha)) = d_{ij}(f(\beta))$ for all ij by Lemma 3.1. Hence $f(\alpha)$ and $f(\beta)$ are congruent, and since they have the same first four points that are not collinear, $f(\alpha)$ and $f(\beta)$ coincide with each other. This implies $\alpha = \beta$. □

Lemma 5.3. *Let* $n \geq 5$*, and let* $\mathcal{M}$ *be a set of Möbius invariants that determine the values of all Möbius invariants for every ordered n-point-set in* $\mathbb{R}^2$*. Then, each* i $(1 \leq i \leq n)$ *appears in at least three members of* $\mathcal{M}$*.*

Proof. Suppose that n appears in at most two members of $\mathcal{M}$, say, in only $[nabc], [nijk] \in \mathcal{M}$. Let $\alpha = a_1 a_2 \dots a_{n-1}$ be a fixed $(n-1)$-point-set in $\mathbb{R}^2$ such that $a_1 a_2 a_3$ is a linear triple and the first four points are not collinear. Let us extend α to an ordered n-point-set in $\mathbb{R}^2$ by adding a point so that $[nabc] = s$ and $[nijk] = t$, for some $s, t > 0$. Then, we may choose any point x as the nth point as far as x satisfies that

$$\frac{a_a x}{a_c x} = s \frac{a_a a_b}{a_b a_c}, \quad \frac{a_i x}{a_k x} = t \frac{a_i a_j}{a_j a_k}.$$

These two equations determine two Apollonian circles, and we may assume that s and t are chosen so that these two Apollonian circles intersect in two points. Then, we can get two ordered n-point-sets β and γ as extensions of α. Note that each member of $\mathcal{M}$ has the same value at β and at γ. But β and γ are not congruent since the $n-1$ points in α are not collinear. Therefore, the values of some Möbius invariant ($\notin \mathcal{M}$) takes different values at β and at γ by Lemma 5.2. This is a contradiction. □

Corollary 5.1. $R(2,5) \geq 4$.

Proof. For any three Möbius invariants, one of $1,2,3,4,5$ cannot appear in all of the three. □

We have $R(2,4) = 2$ by Theorem 4.1, and $R(2,5) = 4$ or 5 by Corollary 5.1 and Theorem 5.1. It seems that the set of four Möbius invariants $[4123], [5431], [5142], [5213]$ determine all values of Möbius invariants for any ordered 5-poin-set in $\mathbb{R}^2$, but I could not prove it.

Problem. Determine $R(2,5)$.

A set of n points in $\mathbb{R}^d$ are called *generic* if the dn coordinates of the n points are algebraically independent over the rationals. Among the $\binom{n}{2}$ distances between generic n points in $\mathbb{R}^d$, how many distances are necessary to determine all distances? If $n \leq d+2$, then all $\binom{n}{2}$ distances are necessary. If $n \gg d$, then the necessary number would be very small compared with $\binom{n}{2}$ by the dimensional restriction. However, to find the exact minimum necessary number is a difficult problem, see Connelly [4], or Jackson-Jordan-Szabadka [6]. Let us state the problem more precisely.

Let $n \geq 4$ and $G = (V, E)$ denote a graph with vertex set $V = \{1, 2, 3, \ldots, n\}$. Then the problem is to characterize the graph G that satisfies the following condition:

($\diamond$) For any two ordered sets α, β of generic n points in $\mathbb{R}^d$, $d_{ij}(\alpha) = d_{ij}(\beta)$ $(ij \in E)$ implies that α and β are congruent in order-preserving fashion.

Recently it was proved (Connelly [4], Jackson *et al* [6]) that in the planar case $d = 2$, every graph $G = (V, E)$ satisfying the condition ($\diamond$) is obtained from the complete graph K_4 by a sequence of Henneberg 1-extension operations and edge additions. The *Henneberg 1-extension operation* on a graph is the following: Remove an edge xy from the graph and add a new vertex z and new edges zx, zy, zw, for some vertex w of the graph other than x, y. Thus, in the planar case $d = 2$, the minimum cardinality of E is $2n - 2$.

Theorem 5.2 (Connelly [4] and Jackson *et al* [6]). *For $n \geq 4$, the minimum number of distances necessary to determine all other distances among generic n points in the plane is $2n - 2$.* □

Put $F_4 = \{[4123], [4213]\}$, and for each $n \geq 5$, define F_n inductively in the following way:

$$F_5 = (F_4 \setminus \{[4213]\}) \cup \{[5431], [5142], [5213]\},$$
$$F_6 = (F_5 \setminus \{[5213]\}) \cup \{[6531], [6152], [6213]\},$$
$$F_7 = (F_6 \setminus \{[6213]\}) \cup \{[7631], [7162], [7213]\},$$
$$\ldots$$
$$F_n = (F_{n-1} \setminus \{[(n-1)213]\}) \cup \{[n(n-1)31], [n1(n-1)2], [n213]\}.$$

Then it seems that F_n determines the values of all Möbius invariants for every *generic* ordered n-point-set in $\mathbb{R}^2$, though I could not prove it. Note that $|F_n| = 2(n-4) + 3 = 2n - 6$.

Suppose that $\mathcal{M}$ is a minimal set of Möbius invariants that determine the values of all Möbius invariants for every *generic* ordered n-point-set α_n in $\mathbb{R}^2$. Since $[4123] - [4213] \neq \pm 1$ always holds for a generic ordered n-point-set, the distances $d_{12}, d_{23}, d_{31}, d_{14}$ and $\mathcal{M}$ determine all distances in α_n in $\mathbb{R}^2$ by Lemma 5.2. From the values of the four distances and the values of the members of $\mathcal{M}$, we obtain $4 + |\mathcal{M}|$ equations for unknowns d_{ij}. Since the minimum number of distances in α_n that determine all distances in α_n is $2n-2$ by Theorem 5.2, it would be natural to expect that the number of equations $4+|\mathcal{M}|$ is at least $2n-2$. This suggests that $|\mathcal{M}| \geq 2n-6$.

Conjecture. For a generic ordered n-point-set ($n \geq 4$) in $\mathbb{R}^2$, the minimum number of Möbius invariants necessary to determine the values of all Möbius invariants is equal to $2n-6$.

6 Invariant fractions

Möbius invariant is generalized as follows. By a *segment*, we mean a distance represented by a pair of points. A *segment-product* is a product of a number of segments. For example, $\boldsymbol{ab}\cdot\boldsymbol{cd}\cdot\boldsymbol{ae}$ is a segment-product.

Theorem 6.1. *A fraction of segment-products is invariant under inversions if and only if the following condition holds:*

($*$) *For each point-symbol, the number of times it appears in the numerator is equal to the number of times it appears in the denominator.*

For example, the fraction $(\boldsymbol{ab}\cdot\boldsymbol{ac}\cdot\boldsymbol{de})/(\boldsymbol{ad}\cdot\boldsymbol{ae}\cdot\boldsymbol{bc})$ is invariant under inversions, but the fraction $(\boldsymbol{ab}\cdot\boldsymbol{cd})/(\boldsymbol{bc}\cdot\boldsymbol{de})$ is not.

Proof. Let us show that ($*$) implies that the fraction is invariant under inversions. Instead the general case, we consider, for example, the fraction $(\boldsymbol{ab}\cdot\boldsymbol{ac}\cdot\boldsymbol{de})/(\boldsymbol{ad}\cdot\boldsymbol{ae}\cdot\boldsymbol{bc})$. Let $\boldsymbol{a}',\ldots,\boldsymbol{e}'$ be the images of $\boldsymbol{a},\ldots,\boldsymbol{e}$ by an inversion with center $\boldsymbol{p}$ and radius r. We show that

$$\frac{\boldsymbol{ab}\cdot\boldsymbol{ac}\cdot\boldsymbol{de}}{\boldsymbol{ad}\cdot\boldsymbol{ae}\cdot\boldsymbol{bc}} = \frac{\boldsymbol{a'b'}\cdot\boldsymbol{a'c'}\cdot\boldsymbol{d'e'}}{\boldsymbol{a'd'}\cdot\boldsymbol{a'e'}\cdot\boldsymbol{b'c'}}. \tag{3}$$

By Lemma 2.1, we have

$$\boldsymbol{a'b'} = \frac{r^2\boldsymbol{ab}}{\boldsymbol{pa}\cdot\boldsymbol{pb}},\ \boldsymbol{a'c'} = \frac{r^2\boldsymbol{ac}}{\boldsymbol{pa}\cdot\boldsymbol{pc}},\ \boldsymbol{d'e'} = \frac{r^2\boldsymbol{de}}{\boldsymbol{pd}\cdot\boldsymbol{pe}}$$

$$\boldsymbol{a'd'} = \frac{r^2\boldsymbol{ad}}{\boldsymbol{pa}\cdot\boldsymbol{pd}},\ \boldsymbol{a'e'} = \frac{r^2\boldsymbol{ae}}{\boldsymbol{pa}\cdot\boldsymbol{pe}},\ \boldsymbol{b'c'} = \frac{r^2\boldsymbol{bc}}{\boldsymbol{pb}\cdot\boldsymbol{pc}}.$$

If these are substituted in the right hand side fraction of (3), then $1/\boldsymbol{pa}$ will appear in the numerator the same number of times as $\boldsymbol{a}'$

appears in the numerator, and also, $1/pa$ will appear in the denominator the same number of times as a' appears in the denominator. Similar things will happen for $1/pb, 1/pc, 1/pd, 1/pe$ and r^2. Since $(*)$ holds for this fraction, all $1/pa, \ldots, 1/pe$ and r^2 will be cancelled out, and we get the equality (3).

To see the *only if* part, suppose that for some point-symbol, say, x, the number of times it appears in the numerator is not equal to the number of times it appears in the denominator. Then we cannot cancel out $1/px$, and the fraction would not be invariant under inversions. □

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