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APPROXIMATION PROCESSES OF FRACTIONAL INTERPOLATION TYPE OPERATORS

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ABSTRACT. We consider approximation processes of fractional interpolation type operators on spaces of functions taking values in normed linear spaces. Consequently, we obtain a far-reaching generalization of the Bernstein type rational functions on $[0, \infty)$ to the multidimensional case for vector-valued functions, and approximation theorems by them with the rate of convergence in terms of the modulus of continuity of functions to be approximated.

1. Introduction

Let \mathbb{N} denote the set of all natural numbers. Let g be a real-valued continuous function on the closed unit interval $\mathbb{I} = [0, 1]$ of the real line \mathbb{R} and let $n \in \mathbb{N}$. Then n th Bernstein polynomial of g is defined by

$$(1) \quad B_n(g)(t) = \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} g\left(\frac{k}{n}\right) \quad (t \in \mathbb{I}).$$

It is well known that the sequence $\{B_n(g)\}_{n \in \mathbb{N}}$ converges uniformly to g on \mathbb{I} , and the Bernstein polynomials and their generalizations play an important role in approximation theory (see, e.g., [1], [2], [8], [13]).

In view of these concerns, Balázs [3] introduced and studied several approximation properties of the Bernstein type rational functions defined as follows:

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Let f be a real-valued function on $[0, \infty)$ and let $n \in \mathbb{N}$, and define

$$(2) \quad R_n(f; x) = \frac{1}{(1 + a_n x)^n} \sum_{k=0}^n \binom{n}{k} (a_n x)^k f\left(\frac{k}{b_n}\right) \quad (x \in [0, \infty)),$$

where $a = \{a_n\}_{n \in \mathbb{N}}$ and $b = \{b_n\}_{n \in \mathbb{N}}$ are suitably chosen sequences of positive real numbers. To compare (1) and (2), setting

$$q_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k} \quad (t \in \mathbb{I}, k = 0, 1, \dots, n)$$

and

$$r_{n,k}(x) = \binom{n}{k} \frac{(a_n x)^k}{(1 + a_n x)^n} \quad (x \in [0, \infty), k = 0, 1, \dots, n),$$

we have

$$r_{n,k}(x) = q_{n,k}\left(\frac{a_n x}{1 + a_n x}\right) \quad (x \in [0, \infty), k = 0, 1, \dots, n),$$

and so

$$R_n(f; x) = B_n(f|_{\mathbb{I}})\left(\frac{a_n x}{1 + a_n x}\right),$$

where $f|_{\mathbb{I}}$ denotes the restriction of f to \mathbb{I} .

In [4], the estimate of the rate of convergence of $R_n(f; x)$ to $f(x)$ given in [3] is improved by an appropriate choice of a and b when f satisfies some more restrictive conditions. Furthermore, in [20] the saturation problem is discussed for $\{R_n\}_{n \in \mathbb{N}}$ and the uniform approximation problem is considered for R_n -like rational functions defined by

$$(3) \quad R_n(B; a; f; x) = \frac{1}{(1 + a_n x)^n} \sum_{k=0}^n \binom{n}{k} (a_n x)^k f(b_{n,k}) \\ = \sum_{k=0}^n r_{n,k}(x) f(b_{n,k}) \quad (x \in [0, \infty)),$$

where $B = (b_{n,k})_{0 \leq k \leq n (n=1,2,\dots)}$ is a matrix whose entries satisfy

$$0 \leq b_{n,0} < b_{n,1} < b_{n,2} < \dots < b_{n,n},$$

and f is a real-valued continuous function on $[0, \infty)$ for which $\lim_{x \rightarrow \infty} f(x)$ exists. Note that if

$$a_n = 1 \quad (n \in \mathbb{N}),$$

and if

$$b_{n,k} = \frac{k}{n - k + 1} \quad (0 \leq k \leq n, n \in \mathbb{N}),$$

then (3) reduces to

$$(4) \quad L_n(f)(x) := \frac{1}{(1+x)^n} \sum_{k=0}^n \binom{n}{k} x^k f\left(\frac{k}{n-k+1}\right),$$

which was introduced by Bleimann, Butzer and Hahn [5]. In [21], the saturation properties of the sequence $\{L_n\}_{n \in \mathbb{N}}$ is established and it is showed that these operators satisfy an asymptotic relation of the Voronovskaja type, i.e.,

$$\lim_{n \rightarrow \infty} n(L_n(x_0) - f(x_0)) = f''(x_0)x_0(1+x_0)^2$$

if f is a real-valued continuous function on $[0, \infty)$ for which $\lim_{x \rightarrow \infty} f(x)$ exists and the second derivative $f''(x_0)$ exists at a point x_0 .

Let $1 \leq p \leq \infty$ be fixed and let \mathbb{R}^r denote the metric linear space of all r -tuples of real numbers, equipped with the usual metric

$$d_p(x, y) := \begin{cases} (\sum_{i=1}^r |x_i - y_i|^p)^{1/p} & (1 \leq p < \infty) \\ \max\{|x_i - y_i| : 1 \leq i \leq r\} & (p = \infty), \end{cases}$$

$$(x = (x_1, x_2, \dots, x_r), y = (y_1, y_2, \dots, y_r) \in \mathbb{R}^r).$$

The purpose of this paper is to generalize (2) for vector-valued functions on the r -dimensional first hyperquadrant

$$[0, \infty)^r := \{x = (x_1, x_2, \dots, x_r) \in \mathbb{R}^r : x_i \geq 0, i = 1, 2, \dots, r\}$$

and to consider their uniform convergence with rates in terms of the modulus of continuity of functions to be approximated.

2. Convergence theorems

Let (X, d) be a locally compact metric space and let $(E, \|\cdot\|)$ be a normed linear space. Let $B(X, E)$ denote the normed linear space of all E -valued bounded functions on X with the supremum norm $\|\cdot\|_X$. Also, we denote by $C(X, E)$ the linear space consisting of all E -valued continuous functions on X and set $BC(X, E) = B(X, E) \cap C(X, E)$. Let $\{Y_\alpha : \alpha \in D\}$ be a family of finite sets, where D is a directed set and let $\{w_\alpha\}_{\alpha \in D}$ be a net of strictly positive real-valued functions on X . For each $\alpha \in D$, let ξ_α be a mapping of Y_α into X and let χ_α be a real-valued function on $X \times Y_\alpha$. Then we define an interpolation type

operator with the weight function w_α by the form

$$U_\alpha(f)(x) = U_\alpha(E; f; x) = w_\alpha(x) \sum_{k \in Y_\alpha} \chi_\alpha(x, k) f(\xi_\alpha(k))$$

$$(\alpha \in D, f \in C(X, E), x \in X)$$

(cf. [14] - [19]), and set

$$W_\alpha(g)(x) = U_\alpha(\mathbb{R}; g; x) = w_\alpha(x) \sum_{k \in Y_\alpha} \chi_\alpha(x, k) g(\xi_\alpha(k))$$

$$(\alpha \in D, g \in C(X, \mathbb{R}), x \in X).$$

For each $x \in X$ and $q > 0$, we define

$$\mu_\alpha(x; q) = w_\alpha(x) \sum_{k \in Y_\alpha} |\chi_\alpha(x, k)| d^q(x, \xi_\alpha(k)) \quad (\alpha \in D),$$

which is called the q th absolute moment of χ_α at x with respect to the weight function w_α . Also, 1_X stands for the unit function defined by $1_X(x) = 1$ for all $x \in X$.

Let X_0 be a compact subset of X and suppose that

$$\chi_\alpha(x, k) \geq 0 \quad (\alpha \in D)$$

for all $(x, k) \in X_0 \times Y_\alpha$.

Theorem 1. *If*

$$(5) \quad \lim_{\alpha} \|W_\alpha(1_X) - 1_X\|_{X_0} = 0$$

and if

$$(6) \quad \lim_{\alpha} \|\mu_\alpha(\cdot; q)\|_{X_0} = 0$$

for some $q > 0$, then

$$(7) \quad \lim_{\alpha} \|U_\alpha(f) - f\|_{X_0} = 0$$

for all $f \in BC(X, E)$.

Proof. This follows from [15, Corollary 3] (cf. [14, Theorem 2 (b)], [15, Theorem 8], [16, Theorem 1]), which remains true without the completeness of E for interpolation type operators (cf. [17]).

In the rest of this section, let $1 \leq p \leq \infty$ be fixed and let

$$(X, d) = ([0, \infty)^r, d_p).$$

For $i = 1, 2, \dots, r$, p_i denotes the i th coordinate function on \mathbb{R}^r defined by

$$p_i(x) = x_i \quad (x = (x_1, x_2, \dots, x_r) \in \mathbb{R}^r).$$

Then we have

$$\mu_\alpha(x) := \mu_\alpha(x; 2) = w_\alpha(x) \sum_{i=1}^r \sum_{k \in Y_\alpha} |\chi_\alpha(x, k)| (p_i(x) - p_i(\xi_\alpha(k)))^2$$

$$(\alpha \in D, x \in X).$$

Note that

$$\mu_\alpha(x) = \sum_{i=1}^r W_\alpha((p_i(x)1_X - p_i)^2)(x) \quad (\alpha \in D, x \in X_0),$$

and so we obtain the following Korovkin type result, which can be more convenient for applications to the concrete weighted interpolation type operators (cf. [14, Theorem 4], [15, Theorems 5 and 6]). For the background of Korovkin-type approximation theory, we refer to the book of Altomare and Campiti [1], in which an excellent source and vast literatures of this theory can be found (cf. [9], [10], [12]).

Corollary 1. *If (5) holds and if*

$$\lim_{\alpha} \mu_\alpha(x) = 0 \quad \text{uniformly in } x \in X_0,$$

then (7) holds for all $f \in BC(X, E)$. In particular, if (5) holds and if

$$(8) \quad \lim_{\alpha} \|W_\alpha(p_i) - p_i\|_{X_0} = 0 \quad (i = 1, 2, \dots, r)$$

and

$$(9) \quad \lim_{\alpha} \|W_\alpha(p_i^2) - p_i^2\|_{X_0} = 0 \quad (i = 1, 2, \dots, r),$$

then (7) holds for every $f \in BC(X, E)$.

Remark 1. Conditions (5), (8) and (9) imply that

$$\lim_{\alpha} \|W_\alpha(g) - g\|_{X_0} = 0$$

for all $g \in BC(X, \mathbb{R})$ (cf. [15, Theorem 6], [14, Theorem 4 (b)]).

Let $n \in \mathbb{N}$ and define

$$p_{n,k}(s, t) = \binom{n}{k} s^k t^{n-k} \quad (s, t \in \mathbb{R}, k = 0, 1, \dots, n).$$

Lemma 1. *Let $s, t, u \in \mathbb{R}$. Then the following equalities hold :*

$$(10) \quad \sum_{k=1}^n k p_{n,k}(s, t) = ns(s+t)^{n-1}.$$

$$(11) \quad \sum_{k=2}^n k(k-1)p_{n,k}(s, t) = n(n-1)s^2(s+t)^{n-2}.$$

$$(12) \quad \sum_{k=1}^n k^2 p_{n,k}(s, t) = ns(ns+t)(s+t)^{n-2}.$$

$$(13) \quad \sum_{k=0}^n (u-k)^2 p_{n,k}(s, t) \\ = (s+t)^{n-2} (s^2 u^2 + nst + 2su(tu - ns) + (tu - ns)^2).$$

In particular, if $tu = ns$, then (13) reduces to

$$\sum_{k=0}^n (u-k)^2 p_{n,k}(s, t) = u(s+t)^{n-2} (s^2 u + t^2).$$

Proof. (10) and (11) immediately follow from the binomial theorem. (12) follows from (10) and (11). Also, (13) follows from (10) and (12).

Lemma 2. Let $\{m_\alpha\}_{\alpha \in D}$ be a net of positive integers and let $\{\beta_\alpha\}_{\alpha \in D}$ be a net of positive real numbers. Let $\{u_\alpha\}_{\alpha \in D}$ and $\{v_\alpha\}_{\alpha \in D}$ be nets of real-valued functions on $[0, \infty)$ such that $u_\alpha(t) + v_\alpha(t) \neq 0$ for all $\alpha \in D$ and all $t \in [0, \infty)$. We define

$$\varphi_{\alpha,k}(t) = \binom{m_\alpha}{k} u_\alpha^k(t) v_\alpha^{m_\alpha - k}(t) \quad (t \in [0, \infty), k = 0, 1, \dots, m_\alpha).$$

Then the following equalities hold for all $\alpha \in D$ and all $t \in [0, \infty)$:

$$(14) \quad \frac{1}{(u_\alpha(t) + v_\alpha(t))^{m_\alpha}} \sum_{k=0}^{m_\alpha} \varphi_{\alpha,k}(t) = 1.$$

$$(15) \quad \frac{1}{(u_\alpha(t) + v_\alpha(t))^{m_\alpha}} \sum_{k=1}^{m_\alpha} \frac{k}{\beta_\alpha} \varphi_{\alpha,k}(t) = \frac{m_\alpha u_\alpha(t)}{\beta_\alpha (u_\alpha(t) + v_\alpha(t))}.$$

$$(16) \quad \frac{1}{(u_\alpha(t) + v_\alpha(t))^{m_\alpha}} \sum_{k=1}^{m_\alpha} \left(\frac{k}{\beta_\alpha}\right)^2 \varphi_{\alpha,k}(t) \\ = \frac{m_\alpha u_\alpha(t) (m_\alpha u_\alpha(t) + v_\alpha(t))}{\beta_\alpha^2 (u_\alpha(t) + v_\alpha(t))^2}.$$

Proof. (14) immediately follows from the binomial theorem. For (15) and (16), we take $n = m_\alpha$, $s = u_\alpha(t)$ and $v_\alpha(t)$ instead of t in Lemma 1 (10) and (12), respectively.

Lemma 3. *Let $\{m_\alpha\}_{\alpha \in D}$ be a net of positive integers and let $\{\beta_\alpha\}_{\alpha \in D}$ be a net of positive real numbers with $\lim_\alpha \beta_\alpha = +\infty$. Let $\{u_\alpha\}_{\alpha \in D}$ and $\{v_\alpha\}_{\alpha \in D}$ be nets of nonnegative functions in $C[0, \infty) := C([0, \infty), \mathbb{R})$ such that*

$$\inf\{u_\alpha(t) + v_\alpha(t) : t \in [0, \infty)\} > 0$$

for each $\alpha \in D$. We define

$$I_\alpha(t) = I_\alpha(m_\alpha; \beta_\alpha; u_\alpha; v_\alpha; t) = \frac{m_\alpha u_\alpha(t)}{\beta_\alpha(u_\alpha(t) + v_\alpha(t))}$$

$$(\alpha \in D, t \in [0, \infty))$$

and

$$J_\alpha(t) = J_\alpha(m_\alpha; \beta_\alpha; u_\alpha; v_\alpha; t) = \frac{m_\alpha u_\alpha(t)(m_\alpha u_\alpha(t) + v_\alpha(t))}{\beta_\alpha^2(u_\alpha(t) + v_\alpha(t))^2}$$

$$(\alpha \in D, t \in [0, \infty)).$$

Let K be a compact subset of $[0, \infty)$. If

$$\lim_\alpha I_\alpha(t) = t \quad \text{uniformly in } t \in K,$$

then

$$\lim_\alpha J_\alpha(t) = t^2 \quad \text{uniformly in } t \in K.$$

Proof. We have

$$J_\alpha(t) = I_\alpha^2(t) + \frac{I_\alpha(t)v_\alpha(t)}{\beta_\alpha(u_\alpha(t) + v_\alpha(t))} \quad (\alpha \in D, t \in [0, \infty)),$$

from which the desired result follows, immediately.

Lemma 4. *Let $\{m_\alpha\}_{\alpha \in D}$, $\{\beta_\alpha\}_{\alpha \in D}$, $\{u_\alpha\}_{\alpha \in D}$ and $\{v_\alpha\}_{\alpha \in D}$ be as in Lemma 3 such that $\beta_\alpha = o(m_\alpha)$, $v_\alpha(t) > 0$ and $m_\alpha u_\alpha(t) = \beta_\alpha t v_\alpha(t)$ for all $\alpha \in D$ and all $t \in [0, \infty)$. Let K be a compact subset of $[0, \infty)$. Then there hold*

$$(17) \quad \lim_\alpha I_\alpha(t) = t \quad \text{uniformly in } t \in K$$

and

$$(18) \quad \lim_\alpha J_\alpha(t) = t^2 \quad \text{uniformly in } t \in K.$$

Proof. Since

$$u_\alpha(t) = \frac{\beta_\alpha t}{m_\alpha} v_\alpha(t) \quad (\alpha \in D, t \in [0, \infty)),$$

we have

$$I_\alpha(t) = \frac{m_\alpha t}{\beta_\alpha t + m_\alpha} \quad (\alpha \in D, t \in [0, \infty)),$$

which establishes (17). Therefore, (18) holds by Lemma 3.

Let $\{n_{\alpha,i}\}_{\alpha \in D, i=1,2,\dots,r}$, be nets of positive integers and let $\{b_{n_{\alpha,i}}\}_{\alpha \in D, i=1,2,\dots,r}$, be nets of positive real numbers such that

$$\lim_{\alpha} b_{n_{\alpha,i}} = +\infty \quad (i = 1, 2, \dots, r).$$

Let $\{g_{n_{\alpha,i}}\}_{\alpha \in D}$ and $\{h_{n_{\alpha,i}}\}_{\alpha \in D, i=1,2,\dots,r}$, be nets of nonnegative functions in $C[0, \infty)$ such that

$$\inf\{g_{n_{\alpha,i}}(t) + h_{n_{\alpha,i}}(t) : t \in [0, \infty)\} > 0$$

for all $\alpha \in D$ and for $i = 1, 2, \dots, r$. Then we define

$$(19) \quad F_\alpha(f)(x) = F_\alpha(E; f; x) = \prod_{i=1}^r \frac{1}{(g_{n_{\alpha,i}}(x_i) + h_{n_{\alpha,i}}(x_i))^{n_{\alpha,i}}} \\ \times \sum_{k_1=0}^{n_{\alpha,1}} \sum_{k_2=0}^{n_{\alpha,2}} \cdots \sum_{k_r=0}^{n_{\alpha,r}} \prod_{i=1}^r \rho_{n_{\alpha,i}, k_i}(x_i) f\left(\frac{k_1}{b_{n_{\alpha,1}}}, \frac{k_2}{b_{n_{\alpha,2}}}, \dots, \frac{k_r}{b_{n_{\alpha,r}}}\right) \\ (\alpha \in D, f \in C(X, E), x = (x_1, x_2, \dots, x_r) \in X),$$

where

$$\rho_{n_{\alpha,i}, k_i}(x_i) = \binom{n_{\alpha,i}}{k_i} g_{n_{\alpha,i}}^{k_i}(x_i) h_{n_{\alpha,i}}^{n_{\alpha,i}-k_i}(x_i) \\ (\alpha \in D, i = 1, 2, \dots, r).$$

We call $F_\alpha, \alpha \in D$, the fractinonal interpolation type operators on $C(X, E)$.

From now on let $K_i, i = 1, 2, \dots, r$, be compact subsets of $[0, \infty)$ and we set

$$X_0 = \prod_{i=1}^r K_i.$$

Theorem 2. *If*

$$(20) \quad \lim_{\alpha} I_\alpha(n_{\alpha,i}; b_{n_{\alpha,i}}; g_{n_{\alpha,i}}; h_{n_{\alpha,i}}; t) = t \quad \text{uniformly in } t \in K_i$$

for $i = 1, 2, \dots, r$, then

$$(21) \quad \lim_{\alpha} \|F_\alpha(f) - f\|_{X_0} = 0$$

for all $f \in BC(X, E)$.

Proof. We define

$$G_\alpha(g)(x) = F_\alpha(\mathbb{R}; g; x) \quad (\alpha \in D, g \in C(X, \mathbb{R}), x \in X).$$

Then by Lemma 2 (14) and (15), we have

$$G_\alpha(1_X)(x) = 1 \quad (\alpha \in D, x \in X)$$

and

$$G_\alpha(p_i)(x) = I_\alpha(n_{\alpha,i}; b_{n_{\alpha,i}}; g_{n_{\alpha,i}}; h_{n_{\alpha,i}}; x_i) \\ (\alpha \in D, x = (x_1, x_2, \dots, x_r) \in X, i = 1, 2, \dots, r).$$

Therefore, (20), Lemma 2 (16) and Lemma 3 yield that

$$\lim_\alpha \|G_\alpha(p_i) - p_i\|_{X_0} = 0 \quad (i = 1, 2, \dots, r)$$

and

$$\lim_\alpha \|G_\alpha(p_i^2) - p_i^2\|_{X_0} = 0 \quad (i = 1, 2, \dots, r).$$

Thus, the desired result (21) follows by Corollary 1.

Let

$$(22) \quad a_{n_{\alpha,i}} := \frac{b_{n_{\alpha,i}}}{n_{\alpha,i}} \quad (\alpha \in D, i = 1, 2, \dots, r)$$

and we define

$$(23) \quad T_\alpha(f)(x) = T_\alpha(E; f; x) = \prod_{i=1}^r \frac{1}{(1 + a_{n_{\alpha,i}} x_i)^{n_{\alpha,i}}} \\ \times \sum_{k_1=0}^{n_{\alpha,1}} \sum_{k_2=0}^{n_{\alpha,2}} \cdots \sum_{k_r=0}^{n_{\alpha,r}} \prod_{i=1}^r \rho_{n_{\alpha,i}, k_i}(x_i) f\left(\frac{k_1}{b_{n_{\alpha,1}}}, \frac{k_2}{b_{n_{\alpha,2}}}, \dots, \frac{k_r}{b_{n_{\alpha,r}}}\right), \\ (\alpha \in D, f \in C(X, E), x = (x_1, x_2, \dots, x_r) \in X),$$

where

$$\rho_{n_{\alpha,i}, k_i}(x_i) = \binom{n_{\alpha,i}}{k_i} (a_{n_{\alpha,i}} x_i)^{k_i} \\ (\alpha \in D, i = 1, 2, \dots, r).$$

Theorem 3. *If*

$$a_{n_{\alpha,i}} = o(1)$$

for $i = 1, 2, \dots, r$, then

$$\lim_\alpha \|T_\alpha(f) - f\|_{X_0} = 0$$

for all $f \in BC(X, E)$.

Proof. Let

$$h_{n_{\alpha,i}} = 1_{[0,\infty)} \quad (\alpha \in D, i = 1, 2, \dots, r)$$

and

$$g_{n_{\alpha,i}}(t) = a_{n_{\alpha,i}} t h_{n_{\alpha,i}}(t) \quad (\alpha \in D, i = 1, 2, \dots, r)$$

for all $t \in [0, \infty)$. Then (19) reduces to (23). Since

$$b_{n_{\alpha,i}} = o(n_{\alpha,i}) \quad (i = 1, 2, \dots, r)$$

and

$$n_{\alpha,i} g_{n_{\alpha,i}}(t) = b_{n_{\alpha,i}} t h_{n_{\alpha,i}}(t) \quad (\alpha \in D, i = 1, 2, \dots, r)$$

for all $t \in [0, \infty)$, we use Lemma 4 and apply Theorem 2 to $F_{\alpha} = T_{\alpha}$.

Remark 2. (23) generalizes (2) to the r -dimensional Bernstein type rational vector-valued functions. Also, (3) can be extended by the following form to the r -dimensional case for vector-valued functions:

$$(24) \quad R_{\alpha}(f)(x) = R_{\alpha}(E; \mathcal{B}; \mathcal{A}; f; x) = \prod_{i=1}^r \frac{1}{(1 + a_{n_{\alpha,i}} x_i)^{n_{\alpha,i}}} \\ \times \sum_{k_1=0}^{n_{\alpha,1}} \sum_{k_2=0}^{n_{\alpha,2}} \cdots \sum_{k_r=0}^{n_{\alpha,r}} \prod_{i=1}^r \binom{n_{\alpha,i}}{k_i} (a_{n_{\alpha,i}} x_i)^{k_i} f(b_{n_{\alpha,1},k_1}, b_{n_{\alpha,2},k_2}, \dots, b_{n_{\alpha,r},k_r}) \\ (\alpha \in D, f \in C(X, E), x = (x_1, x_2, \dots, x_r) \in X),$$

where

$$\mathcal{A} = \{a_{n_{\alpha,i}} : \alpha \in D, i = 1, 2, \dots, r\}$$

and

$$\mathcal{B} = \{b_{n_{\alpha,i},k_i} : 0 \leq k_i \leq n_{\alpha,i}, \alpha \in D, i = 1, 2, \dots, r\}$$

are families of positive real numbers with

$$0 \leq b_{n_{\alpha,i},0} < b_{n_{\alpha,i},1} < b_{n_{\alpha,i},2} < \cdots < b_{n_{\alpha,i},n_{\alpha,i}} \\ (\alpha \in D, i = 1, 2, \dots, r).$$

In particular, the operator $L_n(f)(x)$ defined by (4) is generalized to the r -dimensional case for vector-valued functions defined as follows:

$$(25) \quad L_{\alpha}(f)(x) = L_{\alpha}(E; f; x) = \prod_{i=1}^r \frac{1}{(1 + x_i)^{n_{\alpha,i}}} \sum_{k_1=0}^{n_{\alpha,1}} \sum_{k_2=0}^{n_{\alpha,2}} \cdots \sum_{k_r=0}^{n_{\alpha,r}} \\ \prod_{i=1}^r \binom{n_{\alpha,i}}{k_i} x_i^{k_i} f\left(\frac{k_1}{n_{\alpha,1} - k_1 + 1}, \frac{k_2}{n_{\alpha,2} - k_2 + 1}, \dots, \frac{k_r}{n_{\alpha,r} - k_r + 1}\right) \\ (\alpha \in D, f \in C(X, E), x = (x_1, x_2, \dots, x_r) \in X).$$

3. Rates of convergence

Let $f \in B(X, E)$ and let $\delta \geq 0$. Then we define

$$\omega(f, \delta) = \omega(d; f, \delta) = \sup\{\|f(x) - f(y)\| : x, y \in X, d(x, y) \leq \delta\},$$

which is called the modulus of continuity of f . Obviously, $\omega(f, \cdot)$ is a monotone increasing function on $[0, \infty)$ and

$$\omega(f, 0) = 0, \quad \omega(f, \delta) \leq 2\|f\|_X.$$

Also, f is uniformly continuous on X if and only if

$$\lim_{\delta \rightarrow +0} \omega(f, \delta) = 0.$$

We give here a quantitative form of Theorem 1, in which we estimate the rate of convergence in terms of the modulus of continuity of f . For this we suppose that for each $\alpha \in D$,

$$\chi_\alpha(x, k) \geq 0$$

for all $(x, k) \in X \times Y_\alpha$ and that there exist constants $C \geq 1$ and $K > 0$ such that

$$(26) \quad \omega(f, \xi\delta) \leq (C + K\xi)\omega(f, \delta)$$

for all $\xi, \delta \geq 0$ and all $f \in B(X, E)$.

For the sufficient condition such that (26) holds with $C = K = 1$, see [17, Lemma 1] (cf. [18, Lemma 2.4]).

Let $\{\epsilon_\alpha\}_{\alpha \in D}$ be a net of positive real numbers.

Theorem 4. *Let $q \geq 1$. Then for all $f \in BC(X, E)$, $x \in X$ and all $\alpha \in D$,*

$$(27) \quad \|U_\alpha(f)(x) - f(x)\| \leq |W_\alpha(1_X)(x) - 1| \|f(x)\| \\ + (CW_\alpha(1_X)(x) + Kc_\alpha(x; q))\omega(f, \epsilon_\alpha),$$

where

$$c_\alpha(x; q) = \min\{\epsilon_\alpha^{-q}\mu_\alpha(x; q), \epsilon_\alpha^{-1}(W_\alpha(1_X)(x))^{1-1/q}\mu_\alpha(x; q)^{1/q}\}.$$

In particular, if $W_\alpha(1_X) = 1_X$ for all $\alpha \in D$, then (27) reduces to

$$\|U_\alpha(f)(x) - f(x)\| \leq (C + Kc_\alpha(x; q))\omega(f, \epsilon_\alpha)$$

and

$$c_\alpha(x; q) = \min\{\epsilon_\alpha^{-q}\mu_\alpha(x; q), \epsilon_\alpha^{-1}\mu_\alpha(x; q)^{1/q}\}.$$

Proof. Let $f \in BC(X, E)$ and $x \in X$. Then for all $\alpha \in D$ we have

$$(28) \quad \|U_\alpha(f)(x) - f(x)\| \leq \left\| w_\alpha(x) \sum_{k \in Y_\alpha} \chi_\alpha(x, k) (f(\xi_\alpha(k)) - f(x)) \right\| \\ + \left\| \left(w_\alpha(x) \sum_{k \in Y_\alpha} \chi_\alpha(x, k) - 1 \right) f(x) \right\| \\ = J_\alpha^{(1)}(x) + J_\alpha^{(2)}(x),$$

say. We have

$$J_\alpha^{(2)}(x) = |W_\alpha(1_X)(x) - 1| \|f(x)\|.$$

Also, in view of [18, Lemma 2.7], we obtain

$$(29) \quad J_\alpha^{(1)}(x) \leq (CW_\alpha(1_X)(x) + Kc_\alpha(x; q, \delta))\omega(f, \delta) \quad (\delta > 0),$$

where

$$c_\alpha(x; q, \delta) = \min\{\delta^{-q}\mu_\alpha(x; q), \delta^{-1}(W_\alpha(1_X)(x))^{1-1/q}\mu_\alpha(x; q)^{1/q}\}.$$

Therefore, putting $\delta = \epsilon_\alpha$ in (29), (28) yields the desired inequality (27).

Let Φ be a nonnegative real-valued function on $X^2 := X \times X$ and we define

$$\mu_\alpha(\Phi; x) = \omega_\alpha(x) \sum_{k \in Y_\alpha} \chi_\alpha(x, k) \Phi(x, \xi_\alpha(k)) \quad (\alpha \in D, x \in X),$$

which is called the Φ -moment of χ_α at x with respect to the weight function w_α (cf. [14] - [19]).

Next we suppose that there exist constants $q \geq 1$ and $\kappa > 0$ such that

$$(30) \quad d^q(x, y) \leq \kappa \Phi(x, y)$$

for all $(x, y) \in X^2$.

Remark 3. If

$$(31) \quad \lim_{\alpha} \mu_\alpha(\Phi; x) = 0 \quad \text{uniformly in } x \in X_0,$$

then (30) gives (6). Therefore, by Theorem 1, (5) and (31) establish that (7) holds for every $f \in BC(X, E)$.

Theorem 5. For all $f \in BC(X, E)$, $x \in X$ and all $\alpha \in D$,

$$(32) \quad \|U_\alpha(f)(x) - f(x)\| \leq |W(1_X)(x) - 1| \|f(x)\| \\ + (CW_\alpha(1_X)(x) + K\zeta_\alpha(\Phi; x; q))\omega(f, \epsilon_\alpha),$$

where

$$\zeta_\alpha(\Phi; x; q)$$

$= \min\{\kappa\epsilon_\alpha^{-q}\mu_\alpha(\Phi; x), \kappa^{1/q}\epsilon_\alpha^{-1}(W_\alpha(1_X)(x))^{1-1/q}\mu_\alpha(\Phi; x)^{1/q}\}.$
 In particular, if $W_\alpha(1_X) = 1_X$ for all $\alpha \in D$, then (32) reduces to

$$\|U_\alpha(f)(x) - f(x)\| \leq (C + K\zeta_\alpha(\Phi; x; q))\omega(f, \epsilon_\alpha)$$

and

$$\zeta_\alpha(\Phi; x; q) = \min\{\kappa\epsilon_\alpha^{-q}\mu_\alpha(\Phi; x), \kappa^{1/q}\epsilon_\alpha^{-1}\mu_\alpha(\Phi; x)^{1/q}\}.$$

Proof. By (30), we have

$$\mu_\alpha(x; q) \leq \kappa\mu_\alpha(\Phi; x) \quad (\alpha \in D, x \in X),$$

which implies $c_\alpha(x; q) \leq \zeta_\alpha(\Phi; x; q)$. Therefore, the desired estimate (32) follows from (27).

In the rest of this section, let $1 \leq p \leq \infty$ be fixed and let

$$(X, d) = ([0, \infty)^r, d_p).$$

For each $q \geq 1$, we set

$$c(p, q, r) = \begin{cases} r^{q/p} & (1 \leq p < \infty, p \neq q) \\ 1 & (1 \leq p < \infty, p = q) \\ 1 & (p = \infty). \end{cases}$$

Theorem 6. For all $f \in BC(X, E)$, $x \in X$ and all $\alpha \in D$,

$$(33) \quad \|U_\alpha(f)(x) - f(x)\| \leq |W_\alpha(1_X)(x) - 1| \|f(x)\| \\ + (W_\alpha(1_X)(x) + \tau_\alpha(x; q))\omega(f, \epsilon_\alpha),$$

where

$$\tau_\alpha(x; q) \\ = \min \left\{ c(p, q, r)\epsilon_\alpha^{-q} \sum_{i=1}^r W_\alpha(|p_i(x)1_X - p_i|^q)(x), \right. \\ \left. c(p, q, r)^{1/q}\epsilon_\alpha^{-1} \left(\sum_{i=1}^r W_\alpha(|p_i(x)1_X - p_i|^q)(x) \right)^{1/q} \right\}.$$

In particular, if $W_\alpha(1_X) = 1_X$ for all $\alpha \in D$, then (33) reduces to

$$\|U_\alpha(f)(x) - f(x)\| \leq (1 + \tau_\alpha(x; q))\omega(f, \epsilon_\alpha).$$

Proof. By [17, Lemma 1 (b)] (cf. [18, Lemma 2.4 (b)]), (26) holds with $C = K = 1$. Also, we have

$$d_p^q(x, y) \leq c(p, q, r) \sum_{i=1}^r |p_i(x) - p_i(y)|^q \quad (x, y \in \mathbb{R}^r).$$

Therefore for all $(x, y) \in X^2$, (30) holds with

$$\kappa = c(p, q, r), \quad \Phi(x, y) = \sum_{i=1}^r |p_i(x) - p_i(y)|^q,$$

and so by Theorem 5, we obtain the desired result.

Corollary 2. For all $f \in BC(X, E)$, $x \in X$ and all $\alpha \in D$,

$$(34) \quad \|U_\alpha(f)(x) - f(x)\| \leq |W_\alpha(1_X)(x) - 1| \|f(x)\| \\ + (W_\alpha(1_X)(x) + \tau_\alpha(x)) \omega(f, \epsilon_\alpha),$$

where

$$\tau_\alpha(x) = \min \{ c(p, r) \epsilon_\alpha^{-2} \mu_\alpha(x), \sqrt{c(p, r)} \epsilon_\alpha^{-1} \sqrt{\mu_\alpha(x)} \}$$

and

$$c(p, r) = \begin{cases} r^{2/p} & (1 \leq p < \infty, p \neq 2) \\ 1 & (p = 2, \infty). \end{cases}$$

In particular, if $W_\alpha(1_X) = 1_X$ for all $\alpha \in D$, then (34) reduces to

$$\|U_\alpha(f)(x) - f(x)\| \leq (1 + \tau_\alpha(x)) \omega(f, \epsilon_\alpha).$$

Concerning the rate of convergence of the net $\{F_\alpha\}_{\alpha \in D}$ of the fractional interpolation type operators defined by (19), we get the following result.

Theorem 7. For all $f \in BC(X, E)$, $x = (x_1, x_2, \dots, x_r) \in X$ and all $\alpha \in D$,

$$(35) \quad \|F_\alpha(f)(x) - f(x)\| \leq (1 + \eta_\alpha(x)) \omega(f, \epsilon_\alpha),$$

where

$$(36) \quad \eta_\alpha(x) = \min \{ c(p, r) \epsilon_\alpha^{-2} \theta_\alpha(x), \sqrt{c(p, r)} \epsilon_\alpha^{-1} \sqrt{\theta_\alpha(x)} \}$$

and

$$\theta_\alpha(x) = \sum_{i=1}^r \frac{1}{b_{n_{\alpha,i}}^2 (g_{n_{\alpha,i}}(x_i) + h_{n_{\alpha,i}}(x_i))^2} \\ \times \left((b_{n_{\alpha,i}} x_i g_{n_{\alpha,i}}(x_i))^2 + n_{\alpha,i} g_{n_{\alpha,i}}(x_i) h_{n_{\alpha,i}}(x_i) \right. \\ \left. + 2b_{n_{\alpha,i}} x_i g_{n_{\alpha,i}}(x_i) (b_{n_{\alpha,i}} x_i h_{n_{\alpha,i}}(x_i) - n_{\alpha,i} g_{n_{\alpha,i}}(x_i)) \right. \\ \left. + (b_{n_{\alpha,i}} x_i h_{n_{\alpha,i}}(x_i) - n_{\alpha,i} g_{n_{\alpha,i}}(x_i))^2 \right).$$

Proof. We use Lemma 1 (13) and apply Corollary 2 to $U_\alpha = F_\alpha$.

Corollary 3. Let $a_{n_{\alpha,i}}$ ($\alpha \in D$, $i = 1, 2, \dots, r$) be as in (22). Then for all $f \in BC(X, E)$, $x = (x_1, x_2, \dots, x_r) \in X$ and all $\alpha \in D$,

$$\|T_{\alpha}(f)(x) - f(x)\| \leq (1 + \eta_{\alpha}(x))\omega(f, \epsilon_{\alpha}),$$

where $\eta_{\alpha}(x)$ is given by (36) and

$$(37) \quad \theta_{\alpha}(x) = \sum_{i=1}^r \frac{a_{n_{\alpha,i}}^2 x_i^4 + x_i/b_{n_{\alpha,i}}}{(1 + a_{n_{\alpha,i}} x_i)^2}.$$

Remark 4. Corollary 3 sharply extends and improves [4, Theorem 1] to the very general settings.

Concerning the rate of convergence of the net $\{R_{\alpha}\}_{\alpha \in D}$ of operators defined by (24), we obtain the following result.

Theorem 8. For all $f \in BC(X, E)$, $x = (x_1, x_2, \dots, x_r) \in X$ and all $\alpha \in D$,

$$\|R_{\alpha}(f)(x) - f(x)\| \leq (1 + \gamma_{\alpha}(x))\omega(f, \epsilon_{\alpha}),$$

where

$$\gamma_{\alpha}(x) = \min\{c(p, r)\epsilon_{\alpha}^{-2}\nu_{\alpha}(x), \sqrt{c(p, r)}\epsilon_{\alpha}^{-1}\sqrt{\nu_{\alpha}(x)}\}$$

and

$$\nu_{\alpha}(x) = \sum_{i=1}^r \sum_{k_i=0}^{n_{\alpha,i}} \binom{n_{\alpha,i}}{k_i} \frac{(a_{n_{\alpha,i}} x_i)^{k_i}}{(1 + a_{n_{\alpha,i}} x_i)^{n_{\alpha,i}}} (x_i - b_{n_{\alpha,i}, k_i})^2.$$

Proof. Apply Corollary 2 to $U_{\alpha} = R_{\alpha}$.

Theorem 9. For all $f \in BC(X, E)$, $x = (x_1, x_2, \dots, x_r) \in X$ and all $\alpha \in D$,

$$\|L_{\alpha}(f)(x) - f(x)\| \leq (1 + \zeta_{\alpha}(x))\omega(f, \epsilon_{\alpha}),$$

where

$$\zeta_{\alpha}(x) = \min\{c(p, r)\epsilon_{\alpha}^{-2}\psi_{\alpha}(x), \sqrt{c(p, r)}\epsilon_{\alpha}^{-1}\sqrt{\psi_{\alpha}(x)}\}$$

and

$$(38) \quad \psi_{\alpha}(x) = \sum_{i=1}^r \sum_{k_i=0}^{n_{\alpha,i}} \binom{n_{\alpha,i}}{k_i} \frac{x_i^{k_i}}{(1 + x_i)^{n_{\alpha,i}}} \left(x_i - \frac{k_i}{n_{\alpha,i} - k_i + 1}\right)^2.$$

Proof. Apply Theorem 8 to $R_{\alpha} = L_{\alpha}$.

Remark 5. By [7, Remark 3] (cf. [11, (6)]), we have the the following more explicit expression for the second (absolute) moment (38) of L_{α} :

$$\psi_{\alpha}(x) = \sum_{i=1}^r \frac{(x_i - n_{\alpha,i})x_i^{n_{\alpha,i}+1}}{(1 + x_i)^{n_{\alpha,i}}} + \frac{x_i^{n_{\alpha,i}+1}}{(1 + x_i)^{n_{\alpha,i}}} \sum_{k_i=2}^{n_{\alpha,i}+1} \binom{n_{\alpha,i}+1}{k_i} \frac{x_i^{1-k_i}}{k_i - 1}.$$

Theorem 10. For all $f \in BC(X)$, $x = (x_1, x_2, \dots, x_r) \in X$ and all $\alpha \in D$,

$$(39) \quad \|L_\alpha(f)(x) - f(x)\| \leq (1 + \kappa_\alpha(x))\omega(f, \epsilon_\alpha),$$

where

$$\kappa_\alpha(x) = \min\{c(p, r)\epsilon_\alpha^{-2}\sigma_\alpha(x), \sqrt{c(p, r)}\epsilon_\alpha^{-1}\sqrt{\sigma_\alpha(x)}\}$$

and

$$\sigma_\alpha(x) = 4 \sum_{i=1}^r \frac{x_i(1+x_i)^2}{n_{\alpha,i}}.$$

Proof. By [11, (6)], we have

$$\psi_\alpha(x) \leq \sum_{i=1}^r \left(\frac{x_i}{n_{\alpha,i}} + \frac{x_i^2(1+x_i)}{n_{\alpha,i}} + \frac{3x_i(1+x_i)^2}{n_{\alpha,i}} \right) \leq \sigma_\alpha(x).$$

Therefore, the desired result follows from Theorem 9.

Remark 6. By putting $\epsilon_\alpha\sqrt{\sigma_\alpha(x)}$ instead of ϵ_α in (39), we get the following inequality for all $f \in BC(X, E)$, $x \in X$ and all $\alpha \in D$:

$$(40) \quad \|L_\alpha(f)(x) - f(x)\| \leq (1 + \min\{c(p, r)\epsilon_\alpha^{-2}, \sqrt{c(p, r)}\epsilon_\alpha^{-1}\}) \\ \times \omega\left(f, 2\epsilon_\alpha\sqrt{\sum_{i=1}^r \frac{x_i(x_i+1)^2}{n_{\alpha,i}}}\right).$$

In particular, if $p = 2, \infty$, then (40) reduces to

$$\|L_\alpha(f)(x) - f(x)\| \leq (1 + \min\{\epsilon_\alpha^{-1}, \epsilon_\alpha^{-2}\}) \\ \times \omega\left(f, 2\epsilon_\alpha\sqrt{\sum_{i=1}^r \frac{x_i(x_i+1)^2}{n_{\alpha,i}}}\right),$$

which generalizes the estimate given by Khan [11, Theorem 1].

Remark 7. We set

$$M(x) = \max\{p_i(x)(1+p_i(x))^2 : i = 1, 2, \dots, r\} \quad (x \in X).$$

Then (40) yields the following estimate for all $f \in BC(X, E)$, $x \in X$ and all $\alpha \in D$:

$$(41) \quad \|L_\alpha(f)(x) - f(x)\| \\ \leq \left(1 + \min\left\{\frac{4c(p, r)M(x)}{\epsilon_\alpha^2}, \frac{2\sqrt{c(p, r)}\sqrt{M(x)}}{\epsilon_\alpha}\right\}\right) \omega\left(f, \epsilon_\alpha\sqrt{\sum_{i=1}^r \frac{1}{n_{\alpha,i}}}\right),$$

which particularly reduces to

$$\begin{aligned} & \|L_\alpha(f)(x) - f(x)\| \\ & \leq \left(1 + \min\left\{\frac{4M(x)}{\epsilon_\alpha^2}, \frac{2\sqrt{M(x)}}{\epsilon_\alpha}\right\}\right)\omega\left(f, \epsilon_\alpha\sqrt{\sum_{i=1}^r \frac{1}{n_{\alpha,i}}}\right) \end{aligned}$$

if $p = 2, \infty$.

Remark 8. If

$$n_{\alpha,i} = n_\alpha \quad (\alpha \in D, i = 1, 2, \dots, r),$$

where $\{n_\alpha\}_{\alpha \in D}$ is a net of natural numbers, then by (41) we obtain the following estimate for all $f \in BC(X, E)$, $x \in X$ and all $\alpha \in D$:

$$\begin{aligned} (42) \quad & \|L_\alpha(f)(x) - f(x)\| \\ & \leq \left(1 + \min\left\{4rc(p, r)M(x), 2\sqrt{rc(p, r)}\sqrt{M(x)}\right\}\right)\omega\left(f, \sqrt{\frac{1}{n_\alpha}}\right). \end{aligned}$$

In particular, if $p = 2, \infty$, then (42) reduces to

$$\|L_\alpha(f)(x) - f(x)\| \leq \left(1 + \min\left\{4rM(x), 2\sqrt{r}\sqrt{M(x)}\right\}\right)\omega\left(f, \sqrt{\frac{1}{n_\alpha}}\right).$$

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