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The flat coordinate system of the rational double point
of E8 type

Mitsuo KATO,* Satoshi WATANABE**

Introduction

The flat coordinate system of a versal deformation of an isolated singularity is introduced by Saito [1]. Its existence is proved for rational double points by Saito [2] and the explicit calculations for A1, D1, E8 types are done by Saito, Yano and Sekiguchi [3]. In Yano [4], various free deformations and their flat coordinate systems are considered. A linear subspace by the flat coordinate system of a versal deformation sometimes gives an example of free deformations.

Here we give the calculation for E8 type singularity performed by a computer (DEC 2020 system at RIMS, Kyoto University, and the language REDUCE 2).

The authors are grateful to Dr. T. Yano for many valuable advices.

§ 1

The rational double point of E8 type is defined by the equation

\[ f(x, y) = (1/5)x^5 + (1/3)y^3 = 0. \]

A versal deformation of its singularity is given by

\[ F(x, y) = f(x, y) - t_1x^3y - t_2x^2y - t_3xy - t_4x^2 - t_5y - t_6x - t_7 = 0. \]

If \( t \)'s are generic, \( F_x = 0, F_y = 0 \) define 8 simple zeros:

\[ \{ F_x = 0, F_y = 0 \} = \{ (a_i, b_i), i = 1, \cdots, 8 \} . \]

Let \( g(x, y) \) be holomorphic and put

\[ \omega = g(x, y) \, dxdy / F_x \cdot F_y. \]

Then we define :

\[ \text{res}_{(a_i, b_i)} \omega = g(a_i, b_i) / \text{Hess}_y (a_i, b_i), \]

\[ \text{Res} \omega = \sum_{i=1}^{8} \text{res}_{(a_i, b_i)} \omega, \]

where \( \text{Hess}_y = F_{xx}F_{yy} - F_{xy} \).

It is easily checked that \( \text{Res} \omega \) is a holomorphic function of \( t \). If \( g \) is a polynomial, \( \omega \) can be extended on \( \mathbb{P}_2 = \{ [x : y : z] \} \) and is holomorphic on \( \mathbb{P}_2 - \{ [a_i : b_i : 1], i = 1, \cdots, 8 \} \cup \{ [0 : 1 : 0] \} \). Since the total sum of residues of \( \omega \) on \( \mathbb{P}_2 \) is zero, we have

\[ \text{Res} \omega = - \text{res} [0 : 1 : 0] \omega. \]

§ 2 Residue pairs \( J(\partial_1, \partial_2) \).

Following Yano [2], we define the residue pairs.

\[ J(\partial_1, \partial_2) = \text{Res} \left[ \partial F / \partial t_1 \cdot \partial F / \partial t_1 dx dy / F_x \cdot F_y \right] \]

* Department of Mathematics, University of the Ryukyus.
** Department of Mathematics, Yamagata University.
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\[ \frac{\partial F}{\partial x} \cdot \frac{\partial F}{\partial y} = \text{res}_{[0:1:0]} \left[ \frac{\partial F}{\partial x} \cdot \frac{\partial F}{\partial t} \right] . \]

If \[ \frac{\partial F}{\partial x} \cdot \frac{\partial F}{\partial y} = x^a y^b, \]
then at \([0:1:0]\)
\[ x^a y^b dxdy / F_x \cdot F_y = z^a dz/x / z^{a+b} \cdot \cdot z^b F_x(x/z, 1/z) \cdot z^3 F_y(x/z, 1/z). \]

The residue at \([0:1:0]\) is calculated in the following way.

\[ 1 = A(x, 0) = t_0 x^2 / (1 - t_0 x^2) \]
\[ A_2(x, 0) = \text{res}_{x=0} z_0 x^a dz / z^{a+b} F_x(x/z, 1/z) \cdot z^3 F_y(x/z, 1/z) \]
\[ \zeta = \zeta + A_2(x, 0) \]
\[ A^{(1)}(x, 0) = \frac{A(x, 0) + A(1 + A(x, 0))}{2} \]

\[ \zeta = \zeta + t_0 x \]

\[ \text{res}_{x=0} z_0 x^a dz / z^{a+b} F_x(x/z, 1/z) \cdot z^3 F_y(x/z, 1/z) \]

\[ \zeta = t_0 x^3 \]

§ 3

We consider the "homogeneous" coordinate transformation

\[ s_i = t_i + \sum_{j+k=1} C_{i,j,k} t_j t_k \]

under the condition

\[ J(\partial_{s_i}, \partial_{s_j}) = \begin{cases} 1, & \text{if } i + j = 16 \\
0, & \text{if } i + j \neq 16. \end{cases} \]

Such a coordinate system \( \{ s_i \} \) exists uniquely (Saito [2]), and is called the flat coordinate system of the versal deformation. \( \{ t_i \} \) is also expressed by \( \{ s_i \} \) "homogeneously".

§ 4 Results

\[ f(i, j) \text{ stands for } J(\partial_{t_i}, \partial_{t_j}). \]

\[ f(15, 1) = f(12, 4) = f(10, 6) = f(9, 7) = 1 \]
\[ f(9, 6) = 3t_6 \]
\[ f(10, 4) = f(7, 7) = 3t_4 \]
\[ f(12, 1) = f(9, 4) = f(7, 6) = 9t_4 \]
\[ f(6, 6) = 2t_4 + 27t_4 \]
\[ f(10, 1) = f(7, 4) = 5t_4 + 27t_4 \]
\[ f(9, 1) = f(6, 4) = 3t_4 + 21t_2 t_4 + 81t_6 \]
\[ f(7, 1) = f(4, 4) = 4t_4 t_2 + 18t_2 t_2 + 2t_2 + 108t_4 t_4 + 243t_6 \]
\[ f(6, 1) = 2t_6 + 15t_2 t_4 + 81t_4 t_2 + 90t_4 t_2 + 378t_6 t_4 + 729t_6 \]
\[ f(4, 1) = 3t_4 t_0 + 12t_4 t_0 + 12t_4 t_0 + 81t_4 t_0 + 30t_4 t_4 + 324t_4 t_6 + 171t_4 t_6 + 729t_6 \]
\[ + 1539t_6 t_6 + 2187t_6. \]
\( J(1, 1) = 6t_1^2t_2 + 2t_4t_0 + 54t_4^2t_0 + 20t_1t_4^6 + 216t_4^5t_0 + t_1^2 + 24t_1t_4t_0 \)
\[ + 252t_1^2t_2 + 1134t_1^2t_0 + 81t_1t_4^5t_0 + 12t_1^3t_0 + 1296t_1^4t_0 \]
\[ + 4374t_1^4t_0 + 134t_1^2t_2^2 + 4239t_1^4t_0^2 + 1895t_1^4t_0^4 + 19683t_1^{14} \]

\[ s_1 = t_1 \]
\[ s_4 = t_4 + 2t_1^4 \]
\[ s_6 = t_6 + 2t_1^2t_4 + (14/3)t_1^6 \]
\[ s_7 = t_7 + 2t_1t_4 + (14/3)t_1^2t_4 + (17/3)t_1^7 \]
\[ s_9 = t_9 + (3/2)t_1^3t_4 + 4t_1^2t_6 + (3/2)t_1^4t_4^2 + (52/5)t_1^8t_4 + (299/30)t_1^9 \]
\[ s_{10} = t_{10} + t_1t_4 + 2t_1^3t_4 + t_1t_6 + (11/2)t_1^4t_4 + 3t_1^2t_4^2 + (44/3)t_1^6t_4 \]
\[ + (572/45)t_1^{10} \]
\[ s_{11} = t_{11} + t_1^2t_4t_0 + (7/3)t_1^3t_4 + 2t_1^2t_6 + (28/5)t_1^5t_4 + t_1^2t_6+ 7t_1^2t_6t_0 \]
\[ + (238/15)t_1^6t_4 + (1/3)t_1^3t_4 + 14t_1^4t_4^2 + (1309/30)t_1^8t_4 + (476/15)t_1^{12} \]
\[ s_{15} = t_{15} + t_1^3t_4 + t_1t_4t_0 + (11/5)t_1^5t_4 + t_1t_6 + 3t_1^2t_4t_0 + (88/15)t_1^6t_4 \]
\[ + (1/2)t_1^7 + 3t_1^2t_4 + (1/2)t_1^4t_4 + 11t_1^4t_4 + t_1t_6 + (77/5)t_1^8t_4 \]
\[ + (11/2)t_1^6t_4 + 3t_1^2t_4t_0 + (17/6)t_1^6t_4 + (2002/45)t_1^8t_4 \]
\[ + (22/3)t_1^9t_4 + (308/5)t_1^7t_4^2 + (5642/45)t_1^{11}t_4 \]
\[ + (16523/225)t_1^{15} \]

\[ s_1 = s_1 \]
\[ s_4 = s_4 - 2s_5^4 \]
\[ s_6 = s_6 - 2s_7s_4 + (6/5)s_1^6 \]
\[ s_7 = s_7 - 2s_7^2s_4 + (3/2)s_1^3s_4 + (19/15)s_1^7 \]
\[ s_9 = s_9 - (3/2)s_9^3s_7 - s_9^3s_4 - (3/2)s_9s_4^2 + (23/5)s_1^5s_4 - (28/15)s_1^9 \]
\[ s_{10} = s_{10} - s_9s_8 - (1/2)s_1^3s_7 - s_9s_4 + (3/2)s_9s_4 + (1/2)s_1^2s_4^2 - (2/15)s_1^6s_4 - (11/45)s_1^{10} \]
\[ s_{11} = s_{11} - s_9^2s_8 - (4/3)s_1^3s_9^2 - 2s_1s_9s_8 + (12/5)s_1^5s_9 - s_9^2 + 2s_1^2s_4s_9 \]
\[ - (7/30)s_1^8s_8 - (1/3)s_9s_4^2 - (7/3)s_9s_4^2 - (107/30)s_9s_4^2 + (82/75)s_1^{12} \]
\[ s_{15} = s_{15} - s_9^3s_8 - s_9s_8s_4 + (4/5)s_9^5s_8 - s_9s_8 + (7/15)s_8s_8 - (1/2)s_9s_1^2 \]
\[ + (1/2)s_9^2s_8s_7 - (1/2)s_9^2s_7 + (5/3)s_9^4s_7 - (29/30)s_9^8s_7 \]
\[ + (1/2)s_9^2s_8^2 + (1/2)s_9^3s_8s_4 - (43/30)s_9^5s_8s_4 + (14/45)s_9^9s_8 \]
\[ + (1/3)s_9^3s_9^3 - (43/45)s_9^7s_9 + (421/450)s_9^{11} - (103/450)s_9^{15} \]

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