The b function of a μ-constant deformation of $x^7 + y^5$. 

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Introduction

The versal deformation of the plane curve singularity \( x^7 + y^5 = 0 \) has 24-dimensional parameter space containing 4-dimensional \( \mu \)-constant subspace. The \( \mu \)-constant deformation is given by

\[
(1/7)x^7 + (1/5)y^5 - t_1 x^3 y^2 - t_2 x^5 y^3 - t_{11} x^5 y^3 = 0.
\]

In this paper we stratify the 4-dimensional parameter space into 7 strata, by the condition that on each stratum the \( b \) function is constant. The 7 strata are

\[
\{ t_1 = 0, t_4 + 6t_5 = 0 \}, \quad \{ t_1 = 0, t_4 + 6t_5 = 0, t_6 = 0 \}, \quad \{ t_1 = t_4 = t_5 = 0, t_11 = 0 \}, \quad \{ t_1 = t_4 = t_5 = 0, t_11 = 0 \}, \quad \{ t_1 = t_4 = t_5 = 0, t_11 = 0 \}.
\]

The idea of calculation is essentially due to Yano [4].

Definitions

\( N \) : the set of natural numbers, \( N_0 = N \cup \{ 0 \} \).

\( O \) : the set of germs of holomorphic functions in a neighborhood of 0 in \( \mathbb{C}^{n+1} \).

\( D = O[\partial / \partial x_0, \ldots, \partial / \partial x_n] \).

\( J_f(s) = \{ P(s) \in D[s] : P(s)f^s = 0 \} \), where \( f \in O \).

\( B_{\partial} = D[\delta] = \mathbb{C}[\partial / \partial x_0, \ldots, \partial / \partial x_n] \delta \), where \( \delta \) is the \( \delta \)-function whose support is \( \{ 0 \} \).

Let \( f \in O \), the unitary generator of the ideal \( \{ b(s) \in \mathbb{C}[s] : P(s)f^{s+1} = b(s)f^s \text{ for some } P(s) \in D[s] \} \) is called the \( b \) function of \( f \) and denoted by \( b_f(s) \).

If \( f(0) = 0 \), \( b_f(0) \) is divided by \( s + 1 \), and we put \( b_f(s) = b_f(0)/(s + 1) \).

Let \( d \) be a degree on \( O \), then we can extend \( d \) on \( D[s] \), \( B_{\partial} \) by the rules

\[
d(\partial / \partial x_i) = -d(x_i), \quad d(\delta) = d(s) = 0.
\]

For \( f \in O \), \( P(s) \in D[s] \), \( f^* \) and \( P^*(s) \) denote the homogeneous parts of \( f \) and \( P(s) \), of the lowest degrees.

\( A_m O = \{ f \in O : d(f) \geq m \} \), \( A_m^* O = \{ f \in O : d(f) > m \} \).

§ 1. Preliminary results

1.1 (Theorem of Brieskorn - Pham) Put \( f(x_0, \ldots, x_n) = x_0^a + \ldots + x_n^a \). The characteristic polynomial of the local monodromy of \( (f^{-1}(0), 0) \) is given by

\[
\triangle(t) = \prod (t - a_0 \alpha_1, \ldots, \alpha_n),
\]

where each \( a_i \) ranges over all \( a_i \)th roots of unity other than 1.

For the proof, see Milnor [2]

1.2. (Malgrange [1])

Assume \( f \) has an isolated singularity, then we have

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(1) \( \{ \text{the eigenvalues of local monodromy} \} = \{ \exp(2\pi \sqrt{-1} \alpha) : b(\alpha) = 0 \} \).

And the degree of \( b(s) \) is not greater than \( \mu \), the Milnor number of \( (f^{-1}(0), 0) \).

1.3. (Kashiwara - Yano)

Assume \( f \) has an isolated singularity. Let \( \alpha \) be a root of \( b(\alpha) \), then there exists \( \triangle \in B_\alpha \) with the following properties:

1.3.1 \( f(\alpha) = 0 \); \( (\partial f / \partial x_i) \triangle = 0, i = 0, \ldots, n \).

1.3.2 For all \( P(s) \in f(\alpha), P(\alpha) \triangle = 0 \).

The converse is also true.

1.4. Let \( f(x, y) = (1/m)x^m + (1/n)y^n, (m,n) = 1, \) and \( F \) be a versal deformation defined by \( F(x, y) = f(x, y) - \sum_i x_i y_i \), where the summation ranges all \( (i, j) \) satisfying \( 0 \leq i \leq m - 2, 0 \leq j \leq n - 2 \) and \( i/m + j/n \geq 1 \). Then for each fixed \( t \), \( F^{-1}(0) \) has the same local monodromy as \( f^{-1}(0) \), and by 1.1, the eigenvalues are \( \exp(2\pi \sqrt{-1} ((i + 1)/m + (j + 1)/n)) ; 0 \leq i \leq m - 2, 0 \leq j \leq n - 2 \).

\( b(s) \) is known to be \( \Pi (s + (i + 1)/m + (j + 1)/n) \), where the product ranges \( 0 \leq i \leq m - 2, 0 \leq j \leq n - 2 \). (Miwa [3]).

1.5. Let

\[ F(x, y) = (1/m)x^m + (1/n)y^n - \sum_{i \geq 0} t x^i y^j \] \text{where} \( (m, n) = 1 \), \( t_0 \neq 0 \), \( r = (a_0/m) + (b_0/n) - 1 > 0 \), and \( a_i \geq a_0, b_i \geq b_0 \) for \( i \geq 0 \).

We assume that \( (a_0, b_0) \) satisfies the following condition:

(*) If \( (p, q) \) satisfies

1.5.1 \( 0 \leq p \leq m - 2, 0 \leq q \leq n - 2, \)

1.5.2 \( r = (p/m) + (q/n) - 1 = kr \) for some natural number \( k \),

then we have \( p \geq a_0, q \geq b_0 \).

Then

1.5.3 \( b(s) = \Pi_{0 \leq i \leq m - 2, 0 \leq j \leq n - 2} (s + (i + 1)/m + (j + 1)/n - e_{i,j}) \)

where \( e_{i,j} = \begin{cases} 1 & \text{if} \ i \geq a_0, j \geq b_0, \\ 0 & \text{otherwise}. \end{cases} \)

We first prove the following lemma.

1.6. Lemma.

Let \( F \) be as in 1.5. Assume \( (u, v) \in N_0 \times N_0 \) satisfies

1.6.1 \( u/m + v/n - 2 = kr \) for some \( k \in N \).

Then we have

1.6.2 \( x^u y^v \in (x^{(u-m+1)/m} y^{v-1}) O + (F_x, F_y)(F, F_x, F_y)O \), for some \( (u, v) \in N_0 \times N_0 \) satisfying

1.6.3 \( u/m + v/n - 2 = (k + 1)r \).

Proof.

By 1.6.1, we may assume, without loss of generality, that \( u \geq m \).

Then

1.6.4 \( x^u y^v \in (x^{(u-m+1)/m} y^{v-1}) F_x, x^{u-m+ao} y^{v+bo} O \).

Since \( (u-m, v) \) satisfies (1.5.2) in 1.5, at least one of the following three conditions holds:

1.6.5 \( v \geq n - 1, \)
\[ (1.6.6) \quad u - m \geq m - 1, \]
\[ (1.6.7) \quad u - m \geq a_0, \quad v \geq b_0. \]

If \((1.6.5)\) holds we have
\[ (1.6.8) \quad x^{u} y^{v} \epsilon (F_x F_y, x^{u+m+2a_0} y^{v+b_0}, x^{u+m+2a_0} y^{v+b_0}) O. \]

If \((1.6.6)\) holds we have
\[ (1.6.9) \quad x^{u} y^{v} \epsilon (F_x^2, x^{u+m+2a_0} y^{v+b_0}, x^{u-2m+2a_0} y^{v+2b_0}) O. \]

If \((1.6.7)\) holds we have
\[ (1.6.10) \quad x^{u} y^{v} \epsilon (F_x \cdot (F - X_0 F), x^{u-m+2a_0} y^{v+b_0}) O, \quad \text{where} \quad X_0 = (1/m) x D_x + (1/n) y D_y. \]

In either case, \((1.6.2)\) holds.

**1.7. Proof of 1.5.**

Let \(F\) be as in 1.5, \(d\) be the degree defined by \(d(x) = 1/m, \) \(d(y) = 1/n, \) and \(X_0\) be the Euler operator: \(X_0 = (1/m) x D_x + (1/n) y D_y.\)

Since \(F\) has an isolated singularity, there exists \(m \epsilon N\) so that
\[ (1.7.1) \quad A_m r O \subset (F_x, F_y) O. \]

\(X_0\) has the property:
\[ (1.7.2) \quad F - X_0 F = \Sigma_{i>0} r_i x^{a_i} y^{b_i}, \quad \text{where} \quad r_i = a_i / m + b_i / n - 1, \quad r_0 = r. \]

Hence \((F - X_0 F)_2 \epsilon (x^{2a_0} y^{2b_0}) O,\) and then by Lemma 1.6, we have
\[ (1.7.3) \quad (F - X_0 F)_2 \epsilon (F_x, F_y), \cdot (F, F_x, F_y) + A_m r O \subset (F_x, F_y) \cdot (F, F_x, F_y). \]

This means that there exists an operator \(P_z(s) \epsilon D[s]\) of the form
\[ (1.7.4) \quad P_z(s) = (s + r - X_0)(s - X_0)^{-1} + (\text{higher terms with respect to } d), \]
so that
\[ (1.7.5) \quad P_z(s) F^s = g(x,y) s F^{s-1}, \quad \text{with} \quad \epsilon g(e(F_{xx}) O A_r O + (F_{xy}) O A_r O + (F_{yy}) O A_r O + (x^a y^b) n A_r O. \]

Hence we have an operator \(P_z(s) \epsilon D[s]\) of the form
\[ (1.7.6) \quad P_z(s) = P_z'(s) + g_0(x,y)(s - X_0) + g_1(x,y) D_X + g_2(x,y) D_y, \]
with \(d(g_0) > 0, \quad d(g_1) > r + d(x), \quad d(g_2) > r + d(y). \)

We also have operators of "order 1" in \(J_r(s) : \)
\[ (1.7.7) \quad P_{11}(s) = x^{m - a_0 - 1}(s - X_0) + (\text{higher terms}), \]
\[ (1.7.8) \quad P_{12}(s) = y^{n - b_0 - 1}(s - X_0) + (\text{higher terms}). \]

Now we determine \(\beta_i(s)\) by 1.3.

Let \(\alpha\) be a root of \(g_i(s),\) and \(P(D_x, D_y) \beta\) be the corresponding element of \(B_{pl}\) stated in 1.3. By (1.3.1), we have
\[ (1.7.9) \quad F_x P \delta = F_y P \delta = (F - X_0 F) P \delta = 0. \]

Taking the homogeneous parts of the lowest degree, we get
\[ (1.7.10) \quad x^{m-a} P^* \delta = y^{n-b} P^* \delta = x^a y^b P^* \delta = 0. \]

This means
\[ (1.7.11) \quad P^* = D_x^p D_y^q \text{ with } 0 \leq p \leq m - 2, \quad 0 \leq q < b \quad \text{or} \quad 0 \leq p < a, \quad 0 \leq q \leq n - 2. \]

By (1.3.2), applied to \(P_{11}, P_{12}, \) and \(P_z,\) we get
\[ (1.7.11) \quad (x^{m-a} - (\alpha - X_0) + \ldots) P \delta = (y^{n-b} - (\alpha - X_0) + \ldots) P \delta = 0, \]
\[ (1.7.12) \quad ((\alpha + r - X_0)(\alpha - X_0) + \ldots) P \delta = 0. \]

Since \(X_0 D_x^p D_y^q \delta = -(p + 1) / m + (q + 1) / n) D_x^p D_y^q \delta,\) these mean...
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(1.7.11) $x^{m-\alpha_0-1}(\alpha + (p+1)/m + (q+1)/n)D_x^pD_y^q\delta$

$= y^{n-\beta_0-1}(\alpha + (p+1)/m + (q+1)/n)D_x^pD_y^q\delta$

$= 0$, 

(1.7.12) $(\alpha + r + (p+1)/m + (q+1)/n)(\alpha + (p+1)/m + (q+1)/n)D_x^pD_y^q\delta = 0$.

If 

(1.7.13) $p \geq m - \alpha_0 - 1$ or $q \geq n - \beta_0 - 1$ then by (1.7.11)*, 

(1.7.14) $\alpha = -((p+1)/m + (q+1)/n)$

If 

(1.7.15) $p < m - \alpha_0 - 1$ and $q < n - \beta_0 - 1$ then by (1.7.12)*, 

(1.7.16) $\alpha = -((p+1)/m + (q+1)/n)$ or $-((p+1)/m + (q+1)/n + r)$

The latter is equal to $-((p+\alpha_0 + 1)/m + (q+\beta_0 + 1)/n - 1)$.

By 1.4. and 1.2. $b_\delta(s)$ has distinct $(m-1)(n-1)$ roots, hence (1.7.14) and (1.7.16) give all the roots. This proves 1.5.

§ 2. Example. $F = (1/7)x^7 + (1/5)y^5 - t(x^3y^3 - t_4x^2y^2 - t_6x^4y^3 - t_{11}x^5y^3)$. 

A 

$t_i \neq 0$ 

2.1.

d is defined by $d(x) = 1/7$, $d(y) = 1/5$, and $X_0 = (1/7)xD_x + (1/5)yD_y$.

It is easily seen that 

(2.1.1) $A_{15}O/(F, F_x, F_y)\cap A_{15}O$ is generated by $x^3y^5$ over $C$.

$A_{15}O = (xF_x, yF_y) \cdot A_{15}O$, and

$x^3y^5 \cdot x^3 \equiv y^5 x^3 \equiv t_3^{-1}(F - X_0)F_x^2F_y^2 \text{ (mod higher terms)}$, hence we have

(2.1.2) $A_{15}O = ((F_x, F_y)\cap A_{15}O) \cdot ((F, F_x, F_y)\cap A_{15}O)$

This assures that there exists an equation:

(2.1.3) $(F - X_0)F_x^2 - (1/35)^2(t_3^3x^2y^2F_xF_y + 3t_3^2x^3F_x + 18t_3^2x^2(F - X_0)F_y + 9t_5^4y^2F_x^2 + \ldots ) = 0$

This together with (2.1.1) imply the existence of an operator $P_2(s)eD[s]$ of the form:

(2.1.4) $P_2(s) = (s + 1/35 - X_0)(s - X_0) - (1/35)^2(t_3^2x^2y^2D_xD_y + 3t_3^2x^3yD_y^2$

$+ 3t_3^2x^2D_xD_y + 18t_3^2x^2(s - X_0)D_x + 9t_5^4y^2D_x^2 + 50t_3^2x^2D_y + \ldots )$

so that we have

(2.1.5) $P_2(s)F^* = -(12/35^2)(t_3 + 6t_5^4)x^3y^3F^* + \ldots$.

2.2.

By (2.1.1) and (2.1.2) we have an operator $P_2(s)e_{F_x}(s)$ of the form:

(2.2.1) $P_2(s) = (s + 4/35 - X_0)(s + 1/35 - X_0)(s - X_0) + \text{ (higher terms)}$.

We also have operators in $f_r(s)$ of "order 2 and 1":

(2.2.2) $P_{21}(s) = xP_2(s) + \ldots$, $P_{22}(s) = yP_2(s) + \ldots$,

(2.2.3) $P_{13}(s) = x(s - X_0)(1/35)(t_3^2x^2y^2D_x + 3t_3^2x^3D_y + 9t_5^4x^2yD_y + \ldots )$,

$P_{12}(s) = y(s - X_0)(1/35)(t_3^2x^2D_y + 3t_3^2y^2D_x + 9t_5^4x^2D_y$

$+ t_5^{-1}(27t_1^4 + 4t_4)x^2(s - X_0) + \ldots )$. 


2.3. Assume $\alpha$ be a root of $b\nu(s)$, and $P(Dx,Dy) \delta$ be the corresponding element of $B_{pt}$ with the properties (1.3.1), (1.3.2). Then

\[(2.3.1) P^* = Dx^2Dy^q, \text{ for } 0 \leq p \leq 5, 0 \leq q \leq 3.\]

2.4. If $p \geq 3$ or $q \geq 1$ then the corresponding root $\alpha$ is equal to

\[-(\frac{p+1}{7} + \frac{q+1}{5}).\]

Proof. Apply (1.3.1) to $P_1(s)$, $P_2(s)$.

2.5. $(p,q)$ cannot be either (3,3), (4,3) or (5,3)

Proof. Use the equality $(F-X_0F)P\delta = 0$.

2.6. $(p,q)$ cannot be (5,2) unless $t_4 + 6t_4^5 = 0$.

Proof. Let $(p,q) = (5,2), \alpha = -(6/7 + 3/5)$. Put

\[(2.6.1) P = Dx^5Dy^4 + a_1Dx^2Dy^4 + a_2Dx^3Dy^3 + \ldots .\]

The coefficient of $Dx^2 \delta$ in $F, P\delta$ is $-(24/35) (a_1 + 15t_4)$.

The coefficient of $Dx^3 \delta$ in $P_1(a)P\delta$ is $-(1/35) (36a_4t_4 - 960t_4)$.

Since all these coefficients must be zero, we get $t_4 + 6t_4^5 = 0$.

2.7. Assume $t_4 + 6t_4^5 \neq 0$

If $p = 1$ or 2 and $q = 0$, then by use of $P_2(s)$, we know that $\alpha$ is equal to

\[(2.7.1) -(\frac{p+1}{7} + \frac{q+1}{5}) \text{ or } -(\frac{p+3+1}{7} + \frac{q+1}{5} + \frac{1}{35}).\]

If $p = q = 0$, then by use of $P_3(s)$, $\alpha$ is equal to

\[(2.7.2) 1/7 + 1/5 \text{ or } 1/7 + 1/5 + 1/35 = 4/7 + 4/5 - 1\]

or $1/7 + 1/5 + 4/35 = 6/7 + 3/5 - 1$.

Sine $(p,q)$ cannot be (3,3), (4,3), (5,3) we find that 2.4. and (2.7.1) and (2.7.2) give all the roots of $b\nu(s)$.

By (2.7.1), (2.7.2) together with 2.4, 2.5, and 2.6, we have

\[(2.7.3) b\nu(s) = \Pi_{0 \leq p \leq 5, 0 \leq q \leq 3} (s + (p+1)/7 + (q+1)/5 - e_{p,q}),\]

where $e_{p,q} = \{1 \text{ if } (p,q) = (3,3), (5,2), (4,3), (5,3),$ $0 \text{ otherwise.}\}$

2.8. Assume $t_4 + 6t_4^5 = 0$.

In this case $P_3(s)$ at (2.1.4) belongs to $J_\nu(s)$. Applying 1.3. to $P_3(s)$, we know that for $0 \leq p \leq 2, q = 0$, $\alpha$ is equal to

\[(2.8.1) -(\frac{p+1}{7} + \frac{q+1}{5}) \text{ or } -(\frac{p+3+1}{7} + \frac{q+1}{5} + 1/35) = -(\frac{p+3+1}{7} + \frac{q+1}{5} + 1/35).\]

We get $b\nu(s)$:

\[(2.8.2) b\nu(s) = \Pi_{0 \leq p \leq 5, 0 \leq q \leq 3} (s + (p+1)/7 + (q+1)/5 - e_{p,q}),\]

where $e_{p,q} = \{1 \text{ if } (p,q) = (3,3), (4,3), (5,3),$ $0 \text{ otherwise.}\}$

\[B \quad t_4 = 0\]

2.9. Assume $t_4 \neq 0$, $t_4 \neq 0$.

We have following operators in $J_\nu(s)$:
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(2.9.1) $Q_6(s) = (s + 6/35 - X_0)(s + 4/35 - X_0)(s - X_0) + \text{higher terms.}$

(2.9.2) $Q_7(s) = y(s + 4/35 - X_0)(s - X_0) + \text{higher terms.}$

(2.9.3) $Q_{11}(s) = y^2(s - X_0) + \text{higher terms.}$

$Q_{12}(s) = (2s - 3h_y)(s - X_0) + \text{higher terms.}$

By these operators we can determine $b_7(s)$:

(2.9.4) $b_7(s) = \prod_{0 \leq p \leq 5, 0 \leq q \leq 3} (s + (p + 1)/7 + (q + 1)/5 - e_{p,q}),$

where

$e_{p,q} = \begin{cases} 1 & \text{if } (p,q) = (5,2), (4,3), (5,3), \\ 0 & \text{otherwise}. \end{cases}$

2.10. Assume $t_k = 0$.

We can apply 1.5.

(2.10.1) If $t_k \neq 0, t_k = 0$ then $e_{p,q} = \begin{cases} 1 & \text{if } (p,q) = (5,2), (5,3) \\ 0 & \text{otherwise}. \end{cases}$

(2.10.3) If $t_k = t = 0, t_k \neq 0$ then $e_{p,q} = \begin{cases} 1 & \text{if } (p,q) = (5,3) \\ 0 & \text{otherwise}. \end{cases}$

(2.10.4) If all $t$'s are zero then by 1.4, all $e_{p,q}$ are zero.

References


