FE formulation and theoretical basis of elastic simulation software package including 3D elasticity

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Abstract

3D FE elastic program was included to my FE software package in 1989, though the theory and FE formulation of this 3D FE elastic program was not described. The functionals for 3D and 2D elasticity are explained in this paper regarding the variational principle that is more sophisticated theoretical basis of FE formulation than the principle of virtual work. The FE formulation of 3D elasiticity is explained here based on the principle of virtual work, because the 3D elastic program including my FE software package was developed using the principle of virtual work.

Introduction

I had developed a three-dimensional elastic finite element program in 1989 (Hayashi, 1989a) though the program has not used for a long time. This is because the power of computer which I could use in my laboratory was too poor to calculate any three dimensional problems. Fortunately as the situation surrounding computer has been improved recently, students of my laboratory are enjoying analyze 3D FE models of several interesting tectonic structures. Although the theory and formulation of 3D FE elastic problem are similar to those of 2D problem, it is necessary to show clearly their logical strictness.

I have insisted in my recent paper "Theoretical basis of FE simulation software package" (Hayashi, 2008) that the FE software package has been continuously improved and revised for over thirty years from 1972. First purpose of the present paper is to describe the variational principle regarding 2D and 3D elasticity offering the explicit form of the functional regarding 2D and 3D elasticity with which we can develop the FE formulation (Lanczos, 1974; Hayashi, 1975; Washizu, 1975;Chung, 1978; Hayashi, 1979; Hayashi and Kizaki, 1979; Fletcher, 1984; Hayashi, 1984; Hayashi, 1989b). I have not written the FE formulation using these functionals here because 3D elastic program in my FE software package was corded based on the principle of virtual work (Hayashi and Kizaki, 1972), not based on the variational principle. Second purpose is to explain the FE formulation of 3D elasticity using the principle of virtual work referring Zienkiewicz (1977), though that of 2D elasticity was described in the former work (Hayashi, 2008). As I have written in my former paper, the readers have to be familiar with the variational method to understand the variational principle for elasticity by referring appendices B and C, and Hayashi (1979, 1989b).

The variational principle for elasticity is written as follows (Hayashi, 1979, 1989b).

The equilibrium equation of elasticity is

$$(\lambda+\mu)(u_{k,k})_{,i}+\mu(u_{i,j})_{,j}+\rho f_i=0$$

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We take the functional for elasticity as $\Pi[u_i]$, and estimate the Euler equation of $\Pi[u_i]$. If the Euler equation is identical to the equilibrium equation, the variational principle "u_i that makes the first variation of $\Pi[u_i]$ be zero satisfies the equilibrium equation of elasticity".
How to derive the equilibrium equation of elasticity is attached in appendix A.
Several variational calculuses that are necessary to estimate the Euler equation from the functional are attached in appendices B and C.

Notations

x_i	Cartesian coordinate
t	time
ρ	density
D	domain
∂D	closed surface surrounding D
u_i	displacement vector
v_i	velocity vector
f_i	body force vector per unit mass
n_i	unit normal vector
σ_{ij}	stress tensor
$\frac{D}{Dt}$	Lagrangian differentiation
δ_{ij}	Kronecker's delta
p	pressure
η	coefficient of viscosity
λ, μ	Lame's constants

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f	certain function
g_i	certain vector

$$\frac{Df}{Dt} = f_{,t} + v_j f_{,j}$$

$$f_{,i} = \frac{\partial f}{\partial x_i} \qquad f_{,t} = \frac{\partial f}{\partial t}$$

$$f_{,x} = \frac{\partial f}{\partial x} \qquad f_{,g_i} = \frac{\partial f}{\partial g_i}$$

$$g_{i,j} = \frac{\partial g_i}{\partial x_j} \qquad g_{i,i} = g_{k,k} = div\, \mathbf{g} \quad (\mathbf{g} \text{ is a vector } g_i)$$

Functional of elasticity in two dimension

The fundamental equations in this case are defined,

$(\lambda+\mu)(u_{k,k})_{,i}+\mu(u_{i,j})_{,j}+\rho f_i=0$ (1)

The subscripts i, j and k change from 1 to 2 because the case is of two dimension. They are expressed in every component as

$(\lambda+\mu)(u_{k,k})_{,1}+\mu(u_{1,k})_{,k}+\rho f_1=0$ (1.1)'

$(\lambda+\mu)(u_{k,k})_{,2}+\mu(u_{2,k})_{,k}+\rho f_2=0$ (1.2)'

It will be easily understood that the functional corresponding to (1)' is taken as

$\Pi[u_i]=\iint_s F(x_i,u_i,u_{i,j})ds$ (2)

where $F(x_i,u_i,u_{i,j})$ is the function as

$$F=U(u_{i,j})-\rho f_i u_i=\frac{\lambda}{2}(u_{1,1}+u_{2,2})^2+\mu\left(u_{1,1}{}^2+\frac{(u_{1,1}+u_{2,2})^2}{2}+u_{2,2}{}^2\right)-\rho f_1 u_1-\rho f_2 u_2$$

in which $U(u_{i,j})$ is called either the energy of elastic strain or the function of elastic strain and is defined

$$U(u_{i,j})=\frac{1}{2}\sigma_{i,j}\ \varepsilon_{i,j}$$

Since in this case F is the function of two variables and two functions, the Euler's equation of $\Pi[u_i]$ is written as

$F_{,u1}-(F_{,u1,1})_{,1}-(F_{,u1,2})_{,2}=0$ (3.1)

$F_{,u2}-(F_{,u2,1})_{,1}-(F_{,u2,2})_{,2}=0$ (3.2)

Hereafter we will prove that the fundamental equations (1)' are identical with the Euler's equation (3). If they are identical, we have to admit that the functional defined by (2) is the real functional of (1)'. In the present case, fortunately, it is possible to calculate the Euler's equation explicitly. The preliminary calculations for (3.1), are

$F_{,u1,1}=\lambda u_{k,k}+2\mu u_{1,1}$

$(F_{,u1,1})_{,1}=\lambda(u_{k,k})_{,1}+2\mu(u_{1,1})_{,1}$

$F_{,u1,2}=\mu(u_{1,2}+u_{2,1})$

$(F_{,u1,2})_{,2}=\mu((u_{1,2})_{,2}+(u_{2,2})_{,1})$

$F_{,u1}=-\rho f_1$

Substituting these results into (3.1), we obtain the explicit form of (3.1) as the function of u_1 and u_2.

$(\lambda+\mu)(u_{k,k})_{,1}+\mu(u_{1,k})_{,k}+\rho f_1=0$ (3.1)'

The preliminary calculations for (3.2) are also accomplished as

$F_{,u2,1}=\mu(u_{1,2}+u_{2,1})$

$(F_{,u2,1})_{,1}=\mu((u_{1,1})_{,2}+(u_{2,1})_{,1})$

$F_{,u2,2}=\lambda u_{k,k}+2\mu u_{2,2}$

$(F_{,u2,2})_{,2}=\lambda((u_{k,k})_{,2}+2\mu(u_{2,2})_{,2}$

$F_{,u2}=-\rho f_2$

Then the explicit equation of (3.2) is

$(\lambda+\mu)(u_{k,k})_{,2}+\mu(u_{2,k})_{,k}+\rho f_2=0$ (3.2)'

This is the demonstration that (3)' is identical to (1)'.

Functional of elasticity in three dimension

Needless to say that to the derivation of the Euler's equation for three dimensional elasticity is quite similar to two dimensional case. The fundamental equations of three dimensional case are same to (1) of two dimension, except where the subscripts i, j and k run from 1 to 3. Expand (1) into every x_i components, so we obtain

$$(\lambda+\mu)(u_{k,k})_{,1}+\mu(u_{1,11}+u_{1,22}+u_{1,33})+\rho f_1=0 \qquad (4.1)$$

$$(\lambda+\mu)(u_{k,k})_{,2}+\mu(u_{2,11}+u_{2,22}+u_{2,33})+\rho f_2=0 \qquad (4.2)$$

$$(\lambda+\mu)(u_{k,k})_{,3}+\mu(u_{3,11}+u_{3,22}+u_{3,33})+\rho f_3=0 \qquad (4.3)$$

The following $\Pi[u_i]$ is interpreted as the functional which corresponds to (4).

$$\Pi[u_i]=\iint_v F(x_i,u_i,u_{i,j})dV \qquad (5)$$

where $F(x_i,u_i,u_{i,j})$ is the function represented by

$$F=U(u_{i,j})-\rho f_i u_i=\frac{\lambda}{2}(\varepsilon_{kk})^2+\mu\varepsilon_{ij}\ \varepsilon_{ij}-\rho f_i u_i$$

Expanding these into the x_i components and taking account of three dimensional case, we obtain

$$F=(\frac{\lambda}{2}+\mu)(u_{1,1}{}^2+u_{2,2}{}^2+u_{3,3}{}^2)+\lambda(u_{2,2}u_{3,3}+u_{3,3}u_{1,1}+u_{1,1}u_{2,2})+\frac{\mu}{2}((u_{2,3}+u_{3,2})^2+(u_{3,1}+u_{1,3})^2+(u_{1,2}+u_{2,1})^2)-\rho(f_1u_1+f_2u_2+f_3u_3)$$

As $F(x_i,u_i,u_{i,j})$ is of three variables and three functions, the Euler's equation of $\Pi[u_i]$ will be written as

$$(F_{,u_{1,1}})_{,1}+(F_{,u_{1,2}})_{,2}+(F_{,u_{1,3}})_{,3}-F_{,u_1}=0 \qquad (6.1)$$

$$(F_{,u_{2,1}})_{,1}+(F_{,u_{2,2}})_{,2}+(F_{,u_{2,3}})_{,3}-F_{,u_2}=0 \qquad (6.2)$$

$$(F_{,u_{3,1}})_{,1}+(F_{,u_{3,2}})_{,2}+(F_{,u_{3,3}})_{,3}-F_{,u_3}=0 \qquad (6.3)$$

Then each partial derivative of (6.1) can be calculated as well as two dimensional case.

$$F_{,u_{1,1}}=(\lambda+2\mu)u_{1,1}+\lambda(u_{3,3}+u_{2,2})$$

$$(F_{,u_{1,1}})_{,1}=\lambda(u_{k,k})_{,1}+2\mu u_{1,11}$$

$$F_{,u_{1,2}}=\mu(u_{1,2}+u_{2,1})$$

$$(F_{,u_{1,2}})_{,2}=\mu(u_{1,22}+u_{2,21})$$

$$F_{,u_{1,3}}=\mu(u_{3,1}+u_{1,3})$$

$$(F_{,u_{1,3}})_{,3}=\mu(u_{3,31}+u_{1,33})$$

$$F_{,u_1}=-\rho f_1$$

From the results, we obtain the following simpler form of (6.1).

$$\lambda(u_{k,k})_{,1}+2\mu u_{1,11}+\mu(u_{1,22}+u_{2,21})+\mu(u_{3,31}+u_{1,33})+\rho f_1=0$$

This is changed into

$$\lambda(u_{k,k})_{,1}+\mu(u_{1,11}+u_{1,22}+u_{1,33}+(u_{k,k})_{,1})+\rho f_1=0$$

Then it becomes

$$(\lambda+\mu)(u_{k,k})_{,1}+\mu(u_{1,11}+u_{1,22}+u_{1,33})+\rho f_1=0 \qquad (6.1)'$$

In a quite similar way, we can derive other remaining equations.

$$(\lambda+\mu)(u_{k,k})_{,2}+\mu(u_{2,11}+u_{2,22}+u_{2,33})+\rho f_2=0 \qquad (6.2)'$$

$$(\lambda+\mu)(u_{k,k})_{,3}+\mu(u_{3,11}+u_{3,22}+u_{3,33})+\rho f_3=0 \qquad (6.3)'$$

Therefore it is now clear that three equations of (6.1)', (6.2)' and (6.3)' are identical to (4).

FE formulation of 3D elastic problem

How to construct the elastic FE formulation for three dimension is described in "The finite element method" written by Zienkiewicz (1977). Below is the brief explanation of FE formulation. Proper element is a

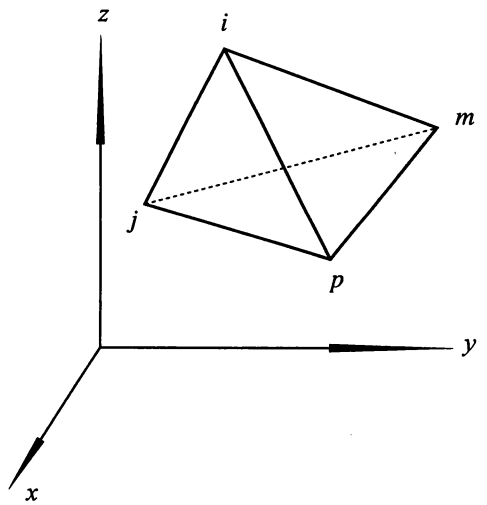

Fig.1 Tetrahedron ijmp

tetrahedral element ijmp shown in Fig.1 when we consider the three dimensional case. Displacement vector is

$$\mathbf{u}=\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

Displacement vector for the point i is

$$\mathbf{a}_i=\begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix}$$

Since the simplest relation is linear, displacement is considered as a linear function of coordinates.

$$u=\alpha_1+\alpha_2 x+\alpha_3 y+\alpha_4 z \qquad (1)$$

Each value of displacement in the points i, j, m and p is written as

$$u_i=\alpha_1+\alpha_2 x_i+\alpha_3 y_i+\alpha_4 z_i$$
$$u_j=\alpha_1+\alpha_2 x_j+\alpha_3 y_j+\alpha_4 z_j$$
$$u_m=\alpha_1+\alpha_2 x_m+\alpha_3 y_m+\alpha_4 z_m$$
$$u_p=\alpha_1+\alpha_2 x_p+\alpha_3 y_p+\alpha_4 z_p$$

Estimating the value of four constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ from the four equations and substituting them into (1), we have u as

$$u=\frac{1}{6V}\{(a_i+b_i x+c_i y+d_i z)u_i-(a_j+b_j x+c_j y+d_j z)u_j+(a_m+b_m x+c_m y+d_m z)u_m-(a_p+b_p x+c_p y+d_p z)u_p\}$$

where

$$6V=\det\begin{bmatrix} 1 & x_i & y_i & z_i \\ 1 & x_j & y_j & z_j \\ 1 & x_m & y_m & z_m \\ 1 & x_p & y_p & z_p \end{bmatrix}$$

and

$$a_i=\det\begin{bmatrix} x_j & y_j & z_j \\ x_m & y_m & z_m \\ x_p & y_p & z_p \end{bmatrix}$$

$$b_i = -\det \begin{bmatrix} 1 & y_j & z_j \\ 1 & y_m & z_m \\ 1 & y_p & z_p \end{bmatrix}$$

$$c_i = -\det \begin{bmatrix} x_j & 1 & z_j \\ x_m & 1 & z_m \\ x_p & 1 & z_p \end{bmatrix}$$

$$d_i = -\det \begin{bmatrix} x_j & y_j & 1 \\ x_m & y_m & 1 \\ x_p & y_p & 1 \end{bmatrix}$$

For this tetrahedral element the element displacement vector $\mathbf{a}^e$ is defined as

$$\mathbf{a}^e = \begin{Bmatrix} \mathbf{a}_i \\ \mathbf{a}_j \\ \mathbf{a}_m \\ \mathbf{a}_p \end{Bmatrix}$$

where

$$\mathbf{a}_i = \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix}$$

Displacement vector **u** is described by the element displacement vector $\mathbf{a}^e$ as

$\mathbf{u}=[\mathbf{I}N_i \ \mathbf{I}N_j \ \mathbf{I}N_m \ \mathbf{I}N_p]\mathbf{a}^e$

where **I** is an identity tensor and

$$N_i = \frac{1}{6V}(a_i + b_i x + c_i y + d_i z)$$

$$N_j = \frac{1}{6V}(a_j + b_j x + c_j y + d_j z)$$

$$N_m = \frac{1}{6V}(a_m + b_m x + c_m y + d_m z)$$

$$N_p = \frac{1}{6V}(a_p + b_p x + c_p y + d_p z)$$

The strain vector **e** is written by **u** as

$$\mathbf{e} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y} \\ \dfrac{\partial w}{\partial x}+\dfrac{\partial u}{\partial z} \end{Bmatrix} = \mathbf{L}\mathbf{u}$$

which is described by the element displacement vector $\mathbf{a}^e$ as

$\mathbf{e} = B\mathbf{a}^e = [\mathbf{B}_i \ \mathbf{B}_j \ \mathbf{B}_m \ \mathbf{B}_p]\mathbf{a}^e$

where

$$\mathbf{B}_i=\begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial z} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x} \end{bmatrix}=\frac{1}{6V}\begin{bmatrix} b_i & 0 & 0 \\ 0 & c_i & 0 \\ 0 & 0 & d_i \\ c_i & b_i & 0 \\ 0 & d_i & c_i \\ d_i & 0 & b_i \end{bmatrix}$$

The stress vector s is shown by the strain vector e as

$$\mathbf{s}=\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}=D\,\mathbf{e}$$

where

$$D=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ & & & & \frac{1-2\nu}{2(1-\nu)} & 0 \\ sym & & & & & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}$$

As well as the case of two dimensions, according to the principle of virtual work, we have the stiffness equation of element.

$$\mathbf{f}^e=K^e\mathbf{u}^e$$

where

$$K_{ij}^{\ e}=B_i^T D B_j V^e$$

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Appendix A

It is discussed here how the equilibrium equations of elasticity

$$(\lambda+\mu)(u_{k,k})_{,i}+\mu(u_{i,j})_{,j}+\rho f_i=0$$

is derived from the conservative law of momentum.

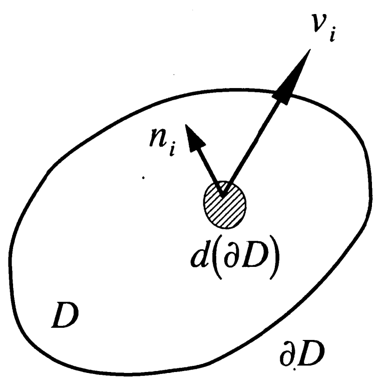

Fig.A1 Conservative law of momentum

(1) Conservative law of momentum

We applied the conservative law of momentum to the mass that occupies a certain domain D and is surrounded with a certain closed surface ∂D as shown in Fig.A1. For any x_i component of momentum, we obtain

$$\frac{\partial}{\partial t}\int_D \rho v_i dD=\int_{\partial D}\sigma_{ij}n_j d(\partial D)-\int_{\partial D}(\rho v_i)v_j n_j d(\partial D)+\int_D \rho f_i dD$$

When we apply the Gauss's theorem to both the first and the second terms of right hand side, the equation becomes

$$\frac{\partial}{\partial t}\int_D \rho v_i dD=\int_D(\sigma_{ij})_{,j}dD-\int_D(\rho v_i v_j)_{,j}dD+\int_D \rho f_i dD$$

Since ∂D is taken arbitrarily, the integrands of both sides must be equal, then

$$(\rho v_i)_{,t}=\sigma_{ij,j}-(\rho v_i v_j)_{,j}+\rho f_i$$

This equation is transformed into

$$(\rho v_i)_{,t} + (\rho v_i v_j)_{,j} = \sigma_{ij,j} + \rho f_i$$

and the left hand side is expanded as follows.

$$\rho_{,t} v_i + \rho v_{i,t} + (\rho v_j)_{,j} v_i + \rho v_j v_{i,j} = \left(\rho_{,t} + (\rho v_j)_{,j}\right) v_i + \rho\left(v_{i,t} + v_j v_{i,j}\right) \qquad \text{(A1)}$$

If we substitute the continuity equation

$$\rho_{,t} + (\rho v_i)_{,i} = 0$$

into (A1), we have

$$\rho\left(v_{i,t} + v_j v_{i,j}\right) \equiv \rho \frac{Dv_i}{Dt}$$

Therefore, we obtain finally the following equation.

$$\rho \frac{Dv_i}{Dt} = \sigma_{ij,j} + \rho f_i \qquad \text{(A2)}$$

This is called the conservative law of momentum for general continuum.

(2) Equilibrium equations of elasticity

The constitutive equation of elasticity is defined as

$$\sigma_{ij} = \lambda u_{k,k} \delta_{ij} + \mu\left(u_{i,j} + u_{j,i}\right)$$

Partially differentiate both sides with respect to x_j, we obtain

$$\sigma_{ij,j} = \lambda (u_{k,k})_{,i} + \mu\left((u_{i,j})_{,j} + (u_{k,k})_{,i}\right) = (\lambda + \mu)(u_{k,k})_{,i} + \mu (u_{i,j})_{,j}$$

Substitute this formula into (A2) and neglect the inertia term, we obtain the equilibrium equation of elasticity

$$(\lambda + \mu)(u_{k,k})_{,i} + \mu (u_{i,j})_{,j} + \rho f_i = 0 \qquad \text{(A3)}$$

In this case, we need not consider the conservative law of mass because number of variables which was handled with is three, that is, u_1, u_2 and u_3, whereas we already have above three equations (if the situation concerned with three dimension).

Appendix B

Typical four cases of Euler's equation are described here.

(1) Euler's equation on one variable and one function

When u is the function of a variable x only, the functional $I[u]$ for a certain function f with variables of $u(x)$ and x is defined as follows.

$$I[u] = \int_{x_1}^{x_2} f(x, u, u')dx$$

The first variation $\delta I[u]$ of the functional $I[u]$ is also defined

$$\delta I[u] = \int_{x_1}^{x_2} df(x, u, u')dx \qquad \text{(B1)}$$

where $df(x,u,u')$ is the total differential of $f(x,u,u')$. The $df(x,u,u')$ is written in more detailed form as follows if we consider the Taylor's theorem.

$df(x,u,u')=\varepsilon(\eta f_{,u}+\eta' f_{,u'})$ (B2)

in which ε is a small positive parameter. Functions $\eta(x)$ and $\eta'(x)$ are continuous within $[x_1, x_2]$ and $\eta(x_1)=\eta(x_2)=0$.

Substitute (B2) into (B1), we obtain

$\delta I[u]=\varepsilon\int_{x_1}^{x_2}(\eta f_{,u}+\eta' f_{,u'})dx$

In order to derive the Euler's equation from $I[u]$, $\delta I[u]$ is taken to be zero, then

$\int_{x_1}^{x_2}(\eta f_{,u}+\eta' f_{,u'})dx=0$

From the technique of the integration of parts, we have

$$\eta f_{,u'}\Big|_{x_1}^{x_2}+\int_{x_1}^{x_2}\eta\left(f_{,u}-\frac{df_{,u}}{dx}\right)dx=0$$

Since $\eta(x_1)=\eta(x_2)=0$ and $\eta f_{,u'}\Big|_{x_1}^{x_2}=0$, the equation is simplified into

$$\int_{x_1}^{x_2}\eta\left(f_{,u}-\frac{df_{,u}}{dx}\right)dx=0$$

Applied the fundamental auxiliary theorem of variational analysis into this equation, the Euler's equation of $I[u]$ is given as

$f_{,u}-\dfrac{df_{,u}}{dx}=0$ (B3)

(2) Euler's equation of one variable and multi-functions

The u_i is only the function of a variable x and the subscript i takes the number 1 to n. The function f depends on the variables of x, $u_i'(x)$, whereas we denoted f as $f(x,u_i,u_i')$. The functional $I[u_i]$ corresponding to $f(x,u_i,u_i')$ is defined as

$I[u_i]=\int_{x_1}^{x_2}f(x,u_i,u_i')dx$

Similar to the former paragraph the first variation of $I[u_i]$ is

$\delta I[u_i]=\int_{x_1}^{x_2}df(x,u_i,u_i')dx$ (B4)

where $df(x,u_i,u_i')$ is the total differential of the $f(x,u_i,u_i')$ and is represented by

$f(x,u_i,u_i')=\sum_i\varepsilon_i(\eta_i f_{,u_i}+\eta_i' f_{,u_i'})$

In this equation ε_i are small real numbers, $\eta_i(x_1)=\eta_i(x_2)=0$ and both the functions of $\eta_i(x)$ and $\eta_i'(x)$ are continuous within $[x_1, x_2]$, therefore (B4) becomes

$\delta I[u_i]=\sum_i\varepsilon_i\int_{x_1}^{x_2}(\eta_i f_{,u_i}+\eta_i' f_{,u_i'})dx$

Let $\delta I[u_i]$ to be zero, the Euler's equation of $I[u_i]$ is obtained

$\int_{x_1}^{x_2}(\eta_i f_{,u_i}+\eta_i' f_{,u_i'})dx=0$

Applying the integration of parts for the equation, we have

$$\eta_i f_{,u_i'}\Big|_{x_1}^{x_2}+\int_{x_1}^{x_2}\eta_i\left(f_{,u_i}-\frac{df_{,u_i}}{dx}\right)dx=0$$

Since from $\eta_i(x_1)=\eta_i(x_2)=0$, the relation $\eta_i f_{,u_i'}\Big|_{x_1}^{x_2}=0$, whereas we have the following equation.

$$\int_{x_1}^{x_2}\eta_i\left(f_{,u_i}-\frac{df_{,u_i}}{dx}\right)dx=0$$

By means of the fundamental auxiliary theorem of the variational analysis, the Euler's equation of $I[u_i]$ is given

$$f_{,u_i} - \frac{df_{,u_i}}{dx} = 0 \qquad \text{(B5)}$$

(3) Euler's equation of multi-variables and one function

The u is depending on n-variables of x_i and f is the function of x_i , u and $u_{,i}$, so that the functional $I[u]$ in respect of $f(x_i,u,u_{,i})$ is written by

$$I[u] = \int_D f(x_i, u, u_{,i}) dD$$

in which D is the n-dimensional spatial domain.

The first variation of $I[u]$ becomes

$$\delta I[u] = \int_D df(x_i, u, u_{,i}) dD$$

where the total differential of $f(x_i,u,u_{,i})$ is

$$df(x_i, u, u_{,i}) = \varepsilon(\eta f_{,u} + \eta_{,1} f_{,u_1} + \eta_{,2} f_{,u_2} + \bullet\bullet\bullet + \eta_{,n} f_{,u_n})$$

By taking that $\delta I[u]=0$, we can give the Euler's equation of $I[u]$ as follows.

$$\int_D (\eta f_{,u} + \eta_{,1} f_{,u_1} + \eta_{,2} f_{,u_2} + \bullet\bullet\bullet + \eta_{,n} f_{,u_n}) dD = 0 \qquad \text{(B6)}$$

The integral of the last n-1 functions on the left side of (B6) is

$$\int_D \left((\eta f_{,u_1})_{,1} + ((\eta f_{,u_2})_{,2} + \bullet\bullet\bullet + (\eta f_{,u_n})_{,n}) \right) dD - \int_D \eta \left((f_{,u_1})_{,1} + (\eta f_{,u_2})_{,2} + \bullet\bullet\bullet + (\eta f_{,u_n})_{,n} \right) dD$$

By using the Gauss's theorem of n-dimension, the first integral of this formula becomes

$$\int_{\partial D} \eta (f_{,u_1} n_1 + f_{,u_2} n_2 + \bullet\bullet\bullet + f_{,u_n} n_n) d(\partial D)$$

Since η (x) vanishes on ∂D, the integral is equal to zero. Therefore the primitive equation (B6) becomes

$$\int_D \eta \left(f_{,u} - (f_{,u_1})_{,1} - (f_{,u_2})_{,2} - \bullet\bullet\bullet - (f_{,u_n})_{,n} \right) dD = 0$$

Apply the fundamental auxiliary theorem of the variational analysis into the equation above, we have the Euler's equation of $I[u]$ as the following form.

$$f_{,u} - (f_{,u_1})_{,1} - (f_{,u_2})_{,2} - \bullet\bullet\bullet - (f_{,u_n})_{,n} = 0 \qquad \text{(B7)}$$

(4) Euler's equation of multi-variables and multi-functions

In the present case u_i is the function of n-variables of x_i and f depends on x_i, u_i and $u_{i,j}$. The functional of $f(x_i,u_i,u_{i,j})$ is defined as

$$I[u_i] = \int_D f(x_i, u_i, u_{i,j}) dD$$

The first variation of $I[u_i]$ is written by

$$\delta I[u_i] = \int_D df(x_i, u_i, u_{i,j}) dD$$

where the total differential of $f(x_i,u_i,u_{i,j})$ is also represented by

$$df(x_i, u_i, u_{i,j}) = \sum_i \varepsilon_i (\eta_i f_{,u_i} + \eta_{i,1} f_{,u_{i,1}} + \eta_{i,2} f_{,u_{i,2}} + \bullet\bullet\bullet + \eta_{i,n} f_{,u_{i,n}})$$

If $\delta I[u_i]$ is replaced to be zero, the Euler's equation of $I[u_i]$ becomes,

$$\int_D (\eta_i f_{,u_i} + \eta_{i,1} f_{,u_{i,1}} + \eta_{i,2} f_{,u_{i,2}} + \bullet\bullet\bullet + \eta_{i,n} f_{,u_{i,n}}) dD = 0 \qquad \text{(B8)}$$

The integral of the left hand side excluding the first term results in

$$\int_D \left((\eta_i f_{,u_{i,1}})_{,1} + (\eta_i f_{,u_{i,2}})_{,2} + \bullet\bullet\bullet + (\eta_i f_{,u_{i,n}})_{,n} \right) dD - \int_D \eta_i \left((f_{,u_{i,1}})_{,1} + (f_{,u_{i,2}})_{,2} + \bullet\bullet\bullet + (f_{,u_{i,n}})_{,n} \right) dD$$

From the Gauss's theorem of n-dimension, the first integral of the formula above is given.

$$\int_{\partial D} \eta_i (f_{,u_{i,1}} n_1 + f_{,u_{i,2}} n_2 + \bullet\bullet\bullet + f_{,u_{i,n}} n_n) d(\partial D)$$

Since the contribution of $\eta_i(x_i)$ vanishes on ∂D, this integral is equal to zero. Consequently, the primitive equation (B8) becomes

$$\int_D \eta_i\left(f_{,u_i} - \left(f_{,u_{i,1}}\right)_{,1} - \left(f_{,u_{i,2}}\right)_{,2} - \bullet\bullet\bullet - \left(f_{,u_{i,n}}\right)_{,n}\right) dD = 0$$

Then we give the last form of the Euler's equation with respect to $I[u_i]$, by means of the fundamental auxiliary theorem of variational analysis.

$$f_{,u_i} - \left(f_{,u_{i,1}}\right)_{,1} - \left(f_{,u_{i,2}}\right)_{,2} - \bullet\bullet\bullet - \left(f_{,u_{i,n}}\right)_{,n} = 0 \qquad \text{(B9)}$$

Appendix C

Special four examples of Euler's equation are described here.

(1) Euler's equation of two variables and two functions

In the case of two variables and two functions the generalized Euler's equation (B9) becomes

$$f_{,u_1} - \left(f_{,u_{1,1}}\right)_{,1} - \left(f_{,u_{1,2}}\right)_{,2} = 0 \qquad \text{(C10.1)}$$

$$f_{,u_2} - \left(f_{,u_{2,1}}\right)_{,1} - \left(f_{,u_{2,2}}\right)_{,2} = 0 \qquad \text{(C10.2)}$$

These are also represented in the detail form as follows.

$$\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial u_{1,1}}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial u_{1,2}}\right) - \frac{\partial f}{\partial u_1} = 0 \qquad \text{(C10.1)'}$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial u_{2,1}}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial u_{2,2}}\right) - \frac{\partial f}{\partial u_2} = 0 \qquad \text{(C10.2)'}$$

(2) Euler's equation of two variables and three functions

$$f_{,u_1} - \left(f_{,u_{1,1}}\right)_{,1} - \left(f_{,u_{1,2}}\right)_{,2} = 0 \qquad \text{(C11.1)}$$

$$f_{,u_2} - \left(f_{,u_{2,1}}\right)_{,1} - \left(f_{,u_{2,2}}\right)_{,2} = 0 \qquad \text{(C11.2)}$$

$$f_{,u_3} - \left(f_{,u_{3,1}}\right)_{,1} - \left(f_{,u_{3,2}}\right)_{,2} = 0 \qquad \text{(C11.3)}$$

These are represented in a detail form by

$$\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial u_{1,1}}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial u_{1,2}}\right) - \frac{\partial f}{\partial u_1} = 0 \qquad \text{(C11.1)'}$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial u_{2,1}}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial u_{2,2}}\right) - \frac{\partial f}{\partial u_2} = 0 \qquad \text{(C11.2)'}$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial u_{3,1}}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial u_{3,2}}\right) - \frac{\partial f}{\partial u_3} = 0 \qquad \text{(C11.3)'}$$

(3) Euler's equation of three variables and three functions

$$f_{,u_1} - \left(f_{,u_{1,1}}\right)_{,1} - \left(f_{,u_{1,2}}\right)_{,2} - \left(f_{,u_{1,3}}\right)_{,3} = 0 \qquad \text{(C12.1)}$$

$$f_{,u_2} - \left(f_{,u_{2,1}}\right)_{,1} - \left(f_{,u_{2,2}}\right)_{,2} - \left(f_{,u_{2,3}}\right)_{,3} = 0 \qquad \text{(C12.2)}$$

$$f_{,u_3} - \left(f_{,u_{3,1}}\right)_{,1} - \left(f_{,u_{3,2}}\right)_{,2} - \left(f_{,u_{3,3}}\right)_{,3} = 0 \qquad \text{(C12.3)}$$

These are also written in a detail form.

$$\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial u_{1,1}}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial u_{1,2}}\right)+\frac{\partial}{\partial x_3}\left(\frac{\partial f}{\partial u_{1,3}}\right)-\frac{\partial f}{\partial u_1}=0 \qquad \text{(C12.1)'}$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial u_{2,1}}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial u_{2,2}}\right)+\frac{\partial}{\partial x_3}\left(\frac{\partial f}{\partial u_{2,3}}\right)-\frac{\partial f}{\partial u_2}=0 \qquad \text{(C12.2)'}$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial u_{3,1}}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial u_{3,2}}\right)+\frac{\partial}{\partial x_3}\left(\frac{\partial f}{\partial u_{3,3}}\right)-\frac{\partial f}{\partial u_3}=0 \qquad \text{(C12.3)'}$$

(4) Euler's equation of three variables and four functions

$$f_{,u_1}-\left(f_{,u_{1,1}}\right)_{,1}-\left(f_{,u_{1,2}}\right)_{,2}-\left(f_{,u_{1,3}}\right)_{,3}=0 \qquad \text{(C13.1)}$$

$$f_{,u_2}-\left(f_{,u_{2,1}}\right)_{,1}-\left(f_{,u_{2,2}}\right)_{,2}-\left(f_{,u_{2,3}}\right)_{,3}=0 \qquad \text{(C13.2)}$$

$$f_{,u_3}-\left(f_{,u_{3,1}}\right)_{,1}-\left(f_{,u_{3,2}}\right)_{,2}-\left(f_{,u_{3,3}}\right)_{,3}=0 \qquad \text{(C13.3)}$$

$$f_{,u_4}-\left(f_{,u_{4,1}}\right)_{,1}-\left(f_{,u_{4,2}}\right)_{,2}-\left(f_{,u_{4,3}}\right)_{,3}=0 \qquad \text{(C13.4)}$$

These are revealed in a detail form as follows.

$$\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial u_{1,1}}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial u_{1,2}}\right)+\frac{\partial}{\partial x_3}\left(\frac{\partial f}{\partial u_{1,3}}\right)-\frac{\partial f}{\partial u_1}=0 \qquad \text{(C13.1)'}$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial u_{2,1}}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial u_{2,2}}\right)+\frac{\partial}{\partial x_3}\left(\frac{\partial f}{\partial u_{2,3}}\right)-\frac{\partial f}{\partial u_2}=0 \qquad \text{(C13.2)'}$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial u_{3,1}}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial u_{3,2}}\right)+\frac{\partial}{\partial x_3}\left(\frac{\partial f}{\partial u_{3,3}}\right)-\frac{\partial f}{\partial u_3}=0 \qquad \text{(C13.3)'}$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial u_{4,1}}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial u_{4,2}}\right)+\frac{\partial}{\partial x_3}\left(\frac{\partial f}{\partial u_{4,3}}\right)-\frac{\partial f}{\partial u_4}=0 \qquad \text{(C13.4)'}$$