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A Note on Moon’s Problem -- Crossings in Random Graphs

Hiroshi Maehara

1. Introduction

Let \( G \) be an (abstract) simple graph. Place the vertices of \( G \) randomly on the surface of a unit sphere \( S \) so that all vertices of \( G \) are distributed independently and uniformly on \( S \). Connect two vertices \( a, b \) by the shortest arc on \( S \) whenever \( \{a, b\} \) is an edge of \( G \). The resulting configuration is called a random drawing of \( G \) on \( S \). A random drawing of \( G \) on a hemisphere \( H \) of \( S \) is defined similarly. The crossing number of a random drawing of \( G \) is the number of pairs of arcs that intersect each other in a point interior to both. (All 'singular' cases of special position may be ignored as they occur with probability zero.)

Moon studied the crossing number \( c(K_n : S) \) in a random drawing of the complete \( n \)-graph \( K_n \) on \( S \). In [2] he stated that the distribution of \( c(K_n : S) \) is asymptotically normal as \( n \) tends to infinity. However the argument to show the asymptotic normality of \( c(K_n : S) \) was incorrect [3].

We show here that the "skewness" of the distribution of \( c(K_n : S) \) tends to a positive constant as \( n \) tends to infinity. Hence the distribution of \( c(K_n : S) \) is never asymptotically normal. On the other hand, it is proved that the distribution of the crossing number \( c(K_n : H) \) in a random drawing of \( K_n \) on a hemisphere \( H \) is asymptotically normal. It is also shown that among all graphs \( G \) with \( n \) vertices and \( m \) edges, the expected value of the crossing number in a random drawing of \( G \) on \( S \) (or \( H \)) takes the largest value when the degrees of the vertices of \( G \) are as equal as possible.

2. Geometric probability on the sphere

We recall here some results on geometric probability on a unit sphere \( S \) for later use (see [4]). For non-antipodal points \( a, b \) of \( S \), \( ab \) denotes the shortest arc (and its length) joining them. A subset \( K \) of \( S \) is convex if \( K \) is hemispherical and \( ab \subset K \) for every non-antipodal \( a, b \) of \( K \).

(2.1) The probability density function of the length \( s = ab \) for two random points \( a, b \) on the unit sphere \( S \) is \( (1/2) \sin s \).

(2.2) The probability that a "random great circle" intersect a convex set \( K \) of perimeter \( L \) is \( L / (2\pi) \).

(2.3) The mean distance between two points on the unit hemisphere \( H \) is \( 4/\pi \).

(2.4) The probability that four random points on the unit hemisphere \( H \) form a convex spherical quadrilateral is \( 3-24/\pi^2 \).

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3. A complete graph on a unit sphere

Consider a random drawing of $K_n$ on $S$ and let $V$ be the vertex set of the drawing. Let $x(abcd)$ be the number of crossings in the six arcs $ab, ac, ad, bc, bd, cd$. Then $x(abcd)$ is a random $(0, 1)$-variable, and the crossing number $c(K_n : S)$ is written as

$$c(K_n : S) = \sum x(abcd),$$

where the summation is taken over all 4-subsets $\{a, b, c, d\}$ of $V$. The conditional probability that $cd$ crosses $ab$ given $ab = s$ follows easily from (2.2):

$$\text{Prob}[cd \text{ crosses } ab \mid ab = s] = s/(4\pi).$$

Then by (2.1) we have the expected values of $x(abcd)$ and $c(K_n : S)$:

$$\mathbb{E}[x(abcd)] = 3/8, \quad \mathbb{E}[c(K_n : S)] = (n/4)(3/8).$$

Three points $a, b, c$, determine three great circles of the sphere $S$, and they divide the surface into eight spherical triangles almost surely: the triangle $T_{abc}$ enclosed by $ab, bc, ca$; the triangles $T_{ab}, T_{bc}, T_{ca}$, each having one side in common with $T_{abc}$; the triangles $T_a, T_b, T_c$, each having one point in common with $T_{abc}$; and the triangle $T$ having no point in common with $T_{abc}$. It is easily seen that the arcs $ab$ and $cd$ intersect each other if and only if the point $d$ is in the triangle $T_{ab}$. Since the probability that $cd$ crosses $ab$ under the condition $ab = s$ is $s/(4\pi)$, we have

$$\mathbb{E}[(area(T_{ab}) \mid ab = s)]/(4\pi) = s/(4\pi),$$

where $\mathbb{E}[\ast \ast]$ denotes the conditional expectation under the condition $\ast \ast$. Since $x(abcd)$ takes the value 1 if and only if $d$ falls in one of $T_{ab}, T_{bc}, T_{ca}$, and since $\text{area}(T_{ab}) = \text{area}(T_c), \ldots, \text{area}(T_{abc}) = \text{area}(T)$, we have

$$\begin{align*}
\mathbb{E}[x(abcd) \mid ab = s] &= \text{Prob}[x(abcd) = 1 \mid ab = s] \\
&= \mathbb{E}[\text{area}(T_{ab}) + \text{area}(T_{bc}) + \text{area}(T_{ca}) \mid ab = s]/(4\pi) \\
&= 1/2 - \mathbb{E}[\text{area}(T_{abc}) \mid ab = s]/(4\pi) = 1/2 - s/(4\pi).
\end{align*}$$

(3.1)

Hence, for different $a, b, c, d, e, f$, we have

$$\begin{align*}
\mathbb{E}[x(abcd) \ x(abef)] &= \mathbb{E}[(1/2 - s/(4\pi))^2] \\
&= (5\pi^2 - 4)/(32\pi^4).
\end{align*}$$

Let $y(abcd) = x(abcd) - 3/8$. Then

$$\mathbb{E}[y(abcd) \ y(cdef)] = (\pi^2 - 8)/(64\pi^4).$$
(Note that \( y(abcdef) \) and \( y(defg) \) are mutually independent as well as \( y(abcd) \) and \( y(efgh) \) are.) Hence the variance of \( c(K_n) \) is

\[
\sigma(n)^2 = E\left[(\Sigma y(abcdef))^2\right] = \left( \frac{n}{4} \right)^2 \left( \frac{n-4}{2} \right)^2 \left( \frac{4}{n^2 - 8} \right) / (64 \pi^2) + O(n^2)
\]

\[
= \left[ (n^2 - 8) / (2^n \pi^2) \right] n^6 + O(n^5).
\]

4. The skewness

We want to estimate the third central moment \( \mu_3(n) \) of \( c(K_n : S) \) when \( n \) is large. First we consider the expected value of the product \( z = x(abcd) x(defg) x(ghia) \). From (3.1) it follows that

\[
E[z \mid ad=s, dg=t, ga=u] = \left[ 1 / (4 \pi) \right]^3 (2\pi - s) (2\pi - t) (2\pi - u)
\]

and hence

\[
E[z] = 7/2^7 - [1/(4\pi)]^3 E[(ad) (dg) (ga)].
\]

Let \( f(s, t, u) \) be the joint probability density function of \( s = ad, t = dg, u = ga \), and let \( f_n(s, t, u) \) be the joint probability density function of \( s, t, u \) when the random three points \( a, d, g \) are chosen independently and uniformly on a fixed hemisphere \( H \) of \( S \). Then

\[
f_n(s, t, u) \text{ Prob } (a, d, g \in H) = f(s, t, u) \text{ Prob } (\Delta adg \cap G = \phi) / 2,
\]

where \( G \) is the great circle bounding \( H \), and \( \text{Prob } (\Delta adg \cap G = \phi) \) is the probability that \( G \) does not cut the triangle \( \Delta adg \) provided that the perimeter of \( \Delta adg \) is \( s + t + u \). Then from (2.2)

\[
\text{Prob } (\Delta adg \cap G = \phi) = 1 - (s + t + u) / (2\pi).
\]

Hence we have

\[
f_n(s, t, u) = 4f(s, t, u) - [2(s + t + u) / \pi] f(s, t, u).
\]

Multiplying both sides by \( (s) (t) = (ad) (dg) \) and integrate (in full range of \( s, t, u \) such that \( s, t, u \) form a spherical triangle), we get

\[
E[(ad) (dg) \mid a, d, g \in H] = 4E[(ad) (dg)] - (4 / \pi) E[(ad)^2 (dg)] - (2 / \pi) E[(ad) (dg) (ga)].
\]

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Since \( E[(ad)(dg)] = E[ad]^2 = (\pi/2)^2 \) and \( E[(ad)^2 dg] = E[(ad)^2] E[dg] = (\pi^3 - 4\pi)/4 \), we have

\[
E[(ad)(dg)(ga)] = 2\pi - (\pi/2) E[(ad)(dg) \mid a, d, d \in H].
\]

On the other hand

\[
E[(ad)(dg) \mid a, d, g \in H] = E[w(d)^2 \mid d \in H],
\]

where \( w(d) = E[(ad) \mid a \in H \text{ with } d \text{ fixed}] \). Since \( w(d) \) is continuous in \( d \in H \) and not constant (because: by (2.3), \( E[w(d) \mid d \in H] = 4/\pi \), however, if \( d \) is the "center" of \( H \) then \( w(d) = 1 \) by (2.1)), we must have

\[
E[w(d)^2 \mid d \in H] > E[w(d) \mid d \in H]^2 = (4/\pi)^2.
\]

Thus we have

\[
E[(ad)(dg)(ga)] < 2\pi - (\pi/2)(4/\pi)^2 = 2\pi - 8/\pi = 3.7367\ldots
\]

\[
< (\pi/2)^3 = 3.8757\ldots
\]

and \( E[z] - (3/8)^3 > 7/2 - [1/(4\pi)]^3 (\pi/2)^3 - (3/8)^3 = 0 \). Hence

\[
p : = E[y(abcd) y(defg) y(ghia)] = E[z] - (3/8)^3 > 0.
\]

Now it is not difficult to see that the third central moment \( \mu_3(n) \) is

\[
\mu_3(n) = E[\{ \Sigma y(abcd) \}^3] = \left( \frac{\pi}{9} \right) \left( \frac{9}{4} \right) \left( \frac{5}{3} \right) (36) p + O(n^8)
\]

\[
= (p/8) n^9 + O(n^8).
\]

Thus the skewness of \( c(K_n:S) \) is

\[
\mu_3(n) / \sigma(n)^3 = (p/8) \left[ 2^9 \pi^2 / (\pi^2 - 8) \right]^{3/2} + o(1),
\]

which tends to a positive constant as \( n \) tends to infinity.

5. A complete graph on a hemisphere

Here we prove the asymptotic normality of the crossing number \( c(K_n:H) \) on a hemisphere \( H \). This is a simple application of a limit lemma proved in \([1]\). First we state the lemma.

Let \( N \) be the set of natural numbers and \( r \) a positive integer. Suppose that for every \( r \)-
element subset $A$ of $N$, there corresponds a random variable $x(A)$ defined on a common probability space and having the same mean $\theta$. We impose the following three conditions.

(5.1) For any finite number of $r$-subsets $A, B, \ldots, D \subset N$, the expected value $E[x(A) \ldots x(D)]$ exists, and for any bijection $\tau : N \to N$, $E[x(\tau A) \ldots x(\tau D)] = E[x(A) \ldots x(D)]$.

(5.2) If $(A \cup \ldots \cup B) \cap (C \cup \ldots \cup D) = \phi$, then

$$E[x(A) \ldots x(D)] = E[x(A) \ldots x(B)] E[x(C) \ldots x(D)].$$

Under the condition (5.1), the covariance $cov[x(A), x(B)]$ of $x(A)$ and $x(B)$ depends only on $|A \cap B|$, the number of elements in $A \cap B$. Let $c(m) = cov[x(A), x(B)]$ if $|A \cap B| = m$. Let $t$ be the minimum value of $m$ such that $c(m) \neq 0$.

(5.3) If $|A \cap (B \cup \ldots \cup D)| \leq t$ and $|A \cap B| < t, \ldots, |A \cap D| < t$, then $E[x(A) \ldots x(D)] = E[x(A)] E[x(B) \ldots x(D)]$.

Note that if $t = 1$ then (5.3) automatically follows from (5.2).

LEMMA. Suppose $x(A)$ (A runs over all $r$-subsets of $N$) satisfy (5.1), (5.2), (5.3), and let $s(n)$ be the sum of $x(A)$ for all $r$-subsets $A$ of $\{1, 2, \ldots, n\}$. Then $[s(n) - \mu]/\sigma$ tends to the normal distribution with zero mean and unit variance as $n$ tends to infinity, where

$$\mu = \binom{n}{r} \theta, \quad \sigma^2 = \left[ c(t) n^{2r-t} \right] / \{t![(r-t)!]^2 \}.$$  

Now we proceed to the proof of asymptotic normality of the distribution of $c(K_n : H)$. Consider a countably infinite number of random points on the unit hemisphere $H$, distributed independently and uniformly on $H$. Label these points by natural numbers. For any 4-subset $A = \{a, b, c, d\}$ of the natural numbers, let $x(A) = x(abcd)$, the number of crossings in six arcs $ab, ac, ad, bc, bd, cd$. Then $x(abcd) = 1$ if four points $a$, $b$, $c$, $d$, form a convex spherical quadrilateral, and $= 0$ otherwise. Thus $\theta = E[x(A)] = 3 - 24/n^2$ by (2.4), and $x(A)$'s clearly satisfy the conditions (5.1), (5.2). Furthermore, $c(K_n : H) = s(n)$, the sum of $x(A)$ for all 4-subsets of $\{1, 2, \ldots, n\}$.

Let $v(a) = E[x(abcd) \mid a : \text{fixed}]$. Then as a function of $a$, $v(a)$ is not constant. This is seen as follows. Suppose $a$ is fixed on the boundary of the hemisphere $H$, and $b, c$ be random points on $H$. Then $H$ is divided by the three great circles determined by $a, b, c$, into six triangles (almost surely) the triangle $T_{abc}$ enclosed by $ab, bc, ca$; the triangles $T_{ab}, T_{bc}, T_{ca}$ each having one side in common with $T_{abc}$; and triangles $T_{ab}, T_{bc}, T_{ca}$ each having one point in common with $T_{abc}$. Further, these six triangles have the same expected area, as easily seen. Since $x(abcd) = 1$ if and only if $d$ falls in $T_{ab}$ or $T_{bc}$ or $T_{ca}$, we have $E[x(abcd)] = v(a) = 1/2 + \theta$. Hence, when $a$ varies in $H$, $v(a)$ also varies, and hence $E[x(abcd) x(aceg)] = E[v(a)^2] > \theta^2$. Thus $c(1) = cov[x(abcd), x(aceg)] > 0$, and hence we can apply the lemma. Therefore, the distribution of $c(K_n : H)$ is asymptotically normal as $n \to \infty$. 

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6. Crossings in a general graph

Let $G$ be a simple graph with $n$ vertices and $m$ edges. Let $V$ be the vertex set of $G$. Denote by $c(G)$ the number of crossings in a random drawing of $G$ on a unit sphere $S$ or on a hemisphere $H$. We show here that the expected value of $c(G)$ is

$$
E[ c(G) ] = (\theta/2) \left[ m^2 + m - \sum_{a \in V} \deg(a)^2 \right].
$$

where $\theta$ is the probability that two non-adjacent arcs $ab$ and $cd$ cross each other, and $\deg(a)$ is the degree of the vertex $a$ of $G$.

Let $\{ a, b \}$ be any edge of $G$. Then there are

$$m - (\deg(a) + \deg(b)) + 1$$

edges not incident to $a$ or $b$. Hence

$$E[ c(G) ] = \sum \left[ m - (\deg(a) + \deg(b)) + 1 \right] \theta/2$$

(where the summation is over all edges $\{a, b\}$ of $G$)

$$= (\theta/2) \left[ m^2 + m - \sum (\deg(a) + \deg(b)) \right].$$

In the summation $\sum (\deg(a) + \deg(b))$, each $\deg(a)$ appears exactly $\deg(a)$ times. Hence

$$\sum (\deg(a) + \deg(b)) = \sum_{a \in V} \deg(a)^2.$$

This proves (6.1).

Let $\bar{d}$ be the average degree of $G$. Then the "variance" of $\deg(a) \ (a \in V)$ is $(\sum \deg(a)^2)/n - (\bar{d})^2$. Therefore, from (6.1) it follows that among all graphs with $n$ vertices and $m$ edges, $E[ c(G) ]$ takes the largest value when $G$ has the minimum variance of $\deg(a)$.

References